CONFIDENTIAL

UNIVERSITI TEKNOLOGI MALAYSIA

FINAL EXAMINATION SEMESTER II
SESSION 2015/2016

COURSE CODE	:	**SKEE / SEE4113**
COURSE NAME	:	**MODERN CONTROL SYSTEM**
LECTURER	:	**PROF. DR. MOHD FUA'AD BIN HJ. RAHMAT**
PROGRAMMES	:	**SKEE / SEE**
SECTION	:	**04**
TIME	:	**2 HOURS 30 MINUTES**
DATE	:	**10 JUNE 2016**
INSTRUCTION	:	**ANSWER FOUR (4) QUESTIONS ONLY.**

THIS EXAMINATION BOOKLET CONSISTS OF 9 PAGES INCLUDING THE FRONT COVER

Question 1

a) The following dynamic equations represent a liquid level control system for two tanks with capacities C_1 and C_2, flow restrictions R_1 and R_2, flow rates $q_i(t)$, $q_1(t)$ and $q_2(t)$, and liquid level heights $h_1(t)$ and $h_2(t)$. $q_i(t)$ is the inlet flow rate and $h_2(t)$ is the output liquid height .

$$q_i(t) - q_1(t) = C_1 \frac{dh_1}{dt} \qquad .....(1.1)$$

$$q_1(t) - q_2(t) = C_2 \frac{dh_2}{dt} \qquad .....(1.2)$$

$$h_1(t) - h_2(t) = R_1 q_1(t) \qquad .....(1.3)$$

$$h_2(t) = R_2 q_2(t) \qquad .....(1.4)$$

i) Represent the system in a state-space representation. *[6 marks]*

ii) Draw a signal flow graph for the system. *[4 marks]*

b) Figure Q1(b) depicts an electromechanical system which consists of an armature controlled DC motor to drive a mechanical rotational load. Let the output be the angular displacement $\theta(t)$.

i) Find the equation of motion for the system. *[8 marks]*

ii) Represent the system in a state space representation. *[7 marks]*

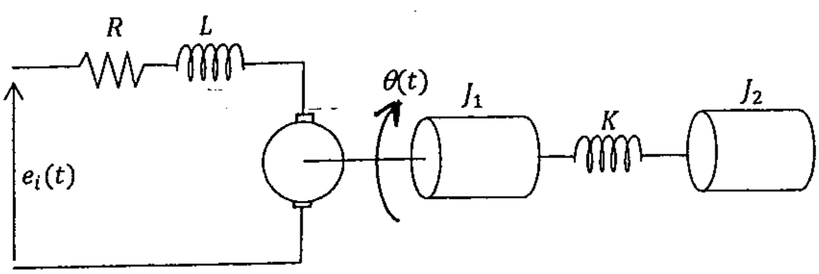


Figure Q1(b)

Question 2

a) Consider a third order system having the following transfer function.

$$\frac{Y(s)}{U(s)} = \frac{4s}{(s+1)(s+2)(s+3)}$$

Obtain a state space representation of the system in observable canonical form.

[8 marks]

b) A system is represented in the following state-space equations.

$$\dot{x}(t) = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 3 \end{bmatrix} x(t)$$

By using similarity transformation matrix ***P*** where *x(t)* = ***P*** *z(t)*, the above state-space equations is transformed into the following equation

$$\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

i) Find the matrix ***P*** *[8 marks]*

ii) Find the corresponding output equation *y(t)*. *[2 marks]*

c) Find the transfer function of the system in part (a) above.

[7 marks]

Question 3

a) A system is represented by the following state-space equation.

$$\dot{x}(t) = \begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 4 \end{bmatrix} x(t)$$

Find the eigenvalues and eigenvector of the system. *[8 marks]*

b) Using the eigenvector obtained as similarity transformation matrix, find the resultant state-space equation of the system.

[7 marks]

c) Using the representation given in part (a) above, solve for *x(t)* and *y(t)* when the input signal is a step function and the initial condition of the system is given by :

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

[10 marks]

Question 4

The state and output equations of a linear time-invariant system are represented by the following equations.

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ -8 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

a) Determine if the above system is controllable and observable. *[4 marks]*

b) Design a linear state feedback controller such that the closed-loop system has 10% overshoot and a settling time of 2 second. *[9 marks]*

c) If a supplementary observer is needed for this system, design an observer such that the settling time is 1 second with equal overshoot response. *[9 marks]*

d) Draw a complete signal flow graph of the system that includes the controller and the observer gains designed above. *[3 marks]*

Question 5

A system is described by the following state equations

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_2(t) + u(t)$$

Given that :

i) The feedback signal is $u(t) = -x_1(t) - k\,x_2(t)$

ii) The initial states are $x_1(0) = 0$, $x_2(0) = 1$

iii) The performance index to be minimized is

$$J = \int_0^{\infty} [x_1^2(t) + x_2^2(t) + u^2(t)]dt$$

a) Show that the matrix P that solves Lyapunov equation is given by

$$P = \begin{bmatrix} \frac{k^2 - 2k + 5}{2(k-1)} & 1 \\ 1 & \frac{k^2 + 3}{2(k-1)} \end{bmatrix}$$

[17 marks]

b) Design the optimal gain to fulfill the performance index. *[8 marks]*

LIST OF FORMULA

1. Time Domain Specifications

First Order System

$$T_r = \frac{2.2}{a}$$

$$T_s = \frac{4}{a}$$

Second Order System

$$\%OS = 100 \; e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

2. State space to transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

3. Similarity Transformation

$$\frac{dx}{dt} = Ax + Bu \quad , \quad y = Cx + Du$$

$$x = Pz$$

$$\frac{dz}{dt} = P^{-1}APz + P^{-1}Bu \quad , \quad y = CPz + Du$$

4. Controllable Canonical Form to Observable Canonical Form

$$A_{OCF} = A_{CCF}^T$$

$$B_{OCF} = C_{CCF}^T$$

$$C_{OCF} = B_{CCF}^T$$

5. State Transition Matrix

$$\Phi(s) = (sI - A)^{-1}$$

6. Time Domain Solution

$$X(s) = (sI - A)^{-1}(x_0 + BU(s))$$

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

7. Controller Design

$$C_M = \begin{bmatrix} B & AB & A^2B & A^{n-1}B \end{bmatrix}$$
$$u(t) = r(t) - Kx(t)$$
$$|sI - (A - BK)| = 0$$

$$z = Px$$
$$P = C_{Mz}C_{Mx}^{-1}$$
$$K_z = K_xP^{-1}$$

8. Observer Design

$$O_M = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{n-1} \end{bmatrix}$$
$$\frac{de_x}{dt} = Ae_x - Le_y$$
$$|sI - (A - LC)| = 0$$

$$z = Px$$
$$P = O_{Mz}^{-1}O_{Mx}$$
$$L_z = PL_x$$

9. Optimal Control

$$J = \int_{t_0}^{t_f} x^T(t)x(t)dt = x^T(t_0)Px(t_0) - x^T(t_\infty)Px(t_\infty)$$

$$J = \int_{t_0}^{t_f} x^T(t)x(t)dt \quad => \quad Q = I$$

$$J = \int_{t_0}^{t_f} u^T(t)u(t)dt \quad => \quad Q = K^T K$$

$$J = \int_{t_0}^{t_f} [x^T(t)x(t) + u^T(t)u(t)]dt \quad => \quad Q = I + K^T K$$

$$H = A + BK$$

$$H^T P + PH = -Q$$

Q1. a) Let h_1 & h_2 be the state variables.

(i)
$$C_1 \frac{dh_1}{dt} = q_i - q_1$$
$$= q_i - \frac{1}{R_1}(h_1 - h_2)$$
$$\therefore \frac{dh_1}{dt} = -\frac{1}{R_1 C_1} h_1 + \frac{1}{R_1 C_1} h_2 + \frac{1}{C_1} q_i$$

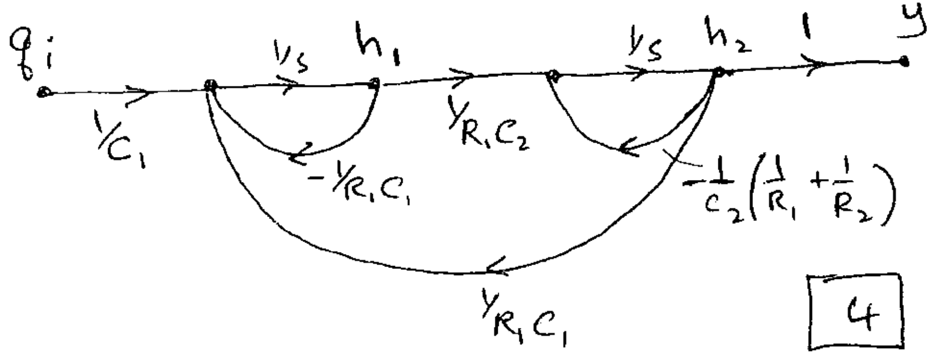
$$C_2 \frac{dh_2}{dt} = q_1 - q_2$$
$$= \frac{1}{R_1}(h_1 - h_2) - \frac{1}{R_2} h_2$$
$$\frac{dh_2}{dt} = \frac{1}{R_1 C_2} h_1 - \frac{1}{C_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) h_2$$

$$\therefore \begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & \frac{1}{R_1 C_1} \\ \frac{1}{R_1 C_2} & -\frac{1}{C_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} q_i$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

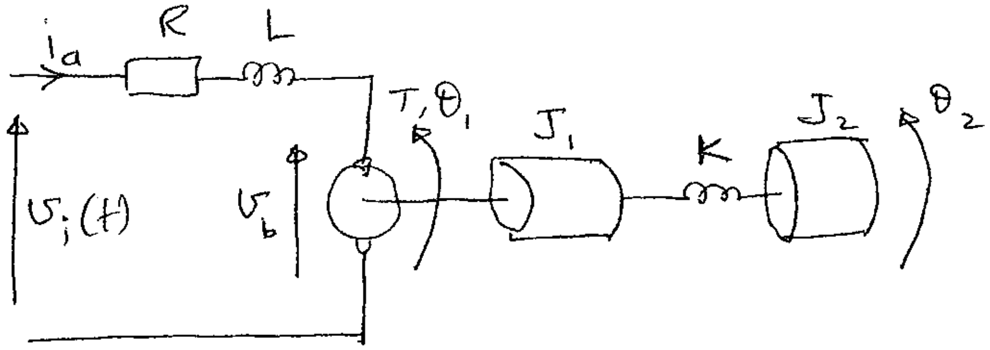
$\boxed{6}$

(ii)



$\boxed{4}$

Q1 b)



$$v_i = i_a R + L \, di_a/dt + v_b \quad \text{——— (1)}$$

$$v_b = K_b \dot{\theta}_1 \quad \text{——— (2)}$$

$$T = K_m i_a = J_1 \ddot{\theta}_1 + K(\theta_1 - \theta_2) \quad \text{——— (3)}$$

$$0 = J_2 \ddot{\theta}_2 + K(\theta_2 - \theta_1) \quad \text{——— (4)}$$

Let

$$x_1 = i_a, \quad x_2 = \theta_1, \quad x_3 = \dot{\theta}_1, \quad x_4 = \theta_2, \quad x_5 = \dot{\theta}_2$$

$$(1): \; L \frac{di_a}{dt} = v_i - R x_1 - K_b x_3 \quad \text{——— (5)}$$

$$\dot{x}_2 = x_3 \quad \text{——— (6)}$$

$$\dot{x}_4 = x_5 \quad \text{——— (7)}$$

$$J_1 \ddot{\theta}_1 = J_1 \dot{x}_3 = K_m x_1 - K x_2 + K x_4 \quad \text{——— (8)}$$

$$J_2 \ddot{\theta}_2 = J_2 \dot{x}_5 = K x_2 - K x_4 \quad \text{——— (9)}$$

Q1

(3)

$$\therefore \quad \dot{x} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{K_b}{L} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{K_m}{J_1} & -\frac{K}{J_1} & 0 & \frac{K}{J_1} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{K}{J_2} & 0 & -\frac{K}{J_2} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} x.$$

Q2 a)

$$\frac{Y}{U} = \frac{4s}{(s+1)(s+2)(s+3)}$$

$$= \frac{4s}{s^3 + 6s^2 + 11s + 6}$$

$$s^3Y + 6s^2Y + 11sY + 6Y = 4sU.$$

$$s^3Y = -6s^2Y + s(\cancel{+11}\,4U - 11Y) - 6Y.$$

$$Y = -\frac{6}{s}Y + \frac{1}{s^2}(4U - 11Y) - \frac{6}{s^3}Y$$

$$= \underbrace{\frac{1}{s}\left[-6Y + \underbrace{\frac{1}{s}\left\{4U - 11Y + \underbrace{\frac{1}{s}(-6Y)}_{X_3}\right\}}_{X_2}\right]}_{X_1}$$

$$Y = X_1 \Rightarrow y = x_1$$

$$X_1 = \frac{1}{s}\left[-6Y + X_2\right] \Rightarrow \dot{x}_1 = -6x_1 + x_2$$

$$X_2 = \frac{1}{s}\left\{4U - 11Y + X_3\right\} \Rightarrow \dot{x}_2 = 4u - 11x_1 + x_3$$

$$X_3 = \frac{1}{s}(-6Y) \Rightarrow \dot{x}_3 = -6x_1$$

$$\dot{x} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$\boxed{8}$

Q2 b)

$$A_x = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}, \quad B_x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C_x = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$A_z = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$.

$$\dot{z} = P^{-1} A_x P z + P^{-1} B_x u$$

$$y = C_x P z.$$

$$\cancel{A_z = P^{-1} A_x P}$$

$$\cancel{P A_z = A_x P}$$

$$B_z = P^{-1} B_x$$

$$P B_z = B_x$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{22} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore P_{12} = -1$$

$$P_{22} = 1$$

$$A_z = P^{-1} A_x P$$

$$P A_z = A_x P$$

$$\begin{bmatrix} P_{11} & \cancel{P_{12}}\,-1 \\ P_{21} & \cancel{P_{22}}\,1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} P_{11} & \cancel{P_{12}}\,-1 \\ P_{21} & \cancel{P_{22}}\,1 \end{bmatrix}$$

Q2:

Element (1,1):

$$3 = -p_{11} + 4p_{21}$$

Element (1,2):

$$p_{11} + 4 = 1 + 4$$

$$\therefore p_{11} = 1$$

$$\therefore p_{21} = 1$$

$$\therefore P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$\boxed{8}$

$$C_z = C_x P$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \end{bmatrix}$$

$$\therefore y = \begin{bmatrix} 4 & 2 \end{bmatrix} z.$$

$\boxed{2}$

c) $$sI - A_x = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} s+1 & -4 \\ 0 & s+3 \end{bmatrix}$$

$$|sI - A| = (s+1)(s+3) = s^2 + 4s + 3$$

$$(sI - A)^{-1} = \frac{1}{s^2+4s+3} \begin{bmatrix} s+3 & 4 \\ 0 & s+1 \end{bmatrix}$$

fm ?

Q2

7

$$C_x (sI - A_x)^{-1} = \frac{1}{s^2+4s+3} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} s+3 & 4 \\ 0 & s+1 \end{bmatrix}$$

$$= \frac{1}{s^2+4s+3} \begin{bmatrix} s+3 & 3s+7 \end{bmatrix}$$

$$G = C_x(sI-A_x)^{-1}B_x = \frac{1}{s^2+4s+3} \begin{bmatrix} s+3 & 3s+7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{2s+4}{s^2+4s+3}$$

$\boxed{7}$

fm 7

Q3

a) $sI - A = \begin{bmatrix} s+4 & 2 \\ -1 & s+1 \end{bmatrix}$

$|sI-A| = (s+4)(s+1) + 2$

$= s^2 + 5s + 6$

$= (s+2)(s+3)$

$\therefore$ Eigenvalue $\lambda_1 = -2$ & $\lambda_2 = -3$

Using $(sI - A)x = 0$ to find eigenvector.

$\begin{bmatrix} s+4 & 2 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = 0.$

$(s+4)x_a + 2x_b = 0$

if $s = \lambda_1 = -2$,

$2x_a + 2x_b = 0$

$\therefore m_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

if $s = \lambda_2 = -3$,

$x_a + 2x_b = 0$

$\therefore m_2 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ or $m_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Q3 (9)

$\therefore$ Eigenvector $M = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ $\boxed{8}$

$$M^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1}A = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -3 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} 2 & 4 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$M^{-1}B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$CM = \begin{bmatrix} 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -4 \end{bmatrix}.$$

$$\therefore \dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} z + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -4 & -4 \end{bmatrix} z.$$

$\boxed{7}$

Q3

c)

$$(sI-A)^{-1} = \frac{1}{\Delta}\begin{bmatrix} s+1 & -2 \\ 1 & s+4 \end{bmatrix}$$

(10)

$$x_0 + BU = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1/s = \begin{bmatrix} 1/s \\ 1 \end{bmatrix}$$

$$X = (sI-A)^{-1}(x_0 + BU)$$

$$= \frac{1}{\Delta}\begin{bmatrix} s+1 & -2 \\ 1 & s+4 \end{bmatrix}\begin{bmatrix} 1/s \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix} \frac{s+1}{s} - 2 \\ 1/s + s + 4 \end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix} \frac{1-s}{s} \\ \frac{s^2+4s+1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+s}{s(s+2)(s+3)} \\ \frac{s^2+4s+1}{s(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} 1/6s - 3/2\,\frac{1}{s+2} + \frac{4}{3}\,\frac{1}{s+3} \\ 1/6s + \frac{3}{2}\,\frac{1}{s+2} - \frac{2}{3}\,\frac{1}{s+3} \end{bmatrix}$$

$$\therefore \quad x = \begin{bmatrix} 1/6 - 3/2\, e^{-2t} + 4/3\, e^{-3t} \\ 1/6 + 3/2\, e^{-2t} - 2/3\, e^{-3t} \end{bmatrix}$$

$\boxed{8}$

$$y = [0 \quad 4]\, x$$

$$= 2/3 + 6e^{-2t} - 8/3\, e^{-3t}$$

$\boxed{2}$

(11)

Q4.

a) $C_m = [B \;\; AB] = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}$

Since $|C_m| \neq 0$, the system is controllable.

$O_m = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Sine O_m is not singular, the system is observable.

$\boxed{4}$

b) $\zeta = \dfrac{-\ln(0.1)}{\sqrt{\pi^2 + \ln^2(0.1)}} = 0.59$

$T_s = \dfrac{4}{\zeta \omega_n} = 2$

$\therefore \omega_n = 3.39$.

<u>Desired CE:</u>

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$s^2 + 2 \times 0.59 \times 3.39 s + (3.39)^2 = 0$

$s^2 + 4s + 11.49 = 0$.

Q4 (12)

$$A-BK = \begin{bmatrix} 1 & 1 \\ -8 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -(8+k_1) & -(5+k_2) \end{bmatrix}$$

$$sI-(A-BK) = \begin{bmatrix} s-1 & -1 \\ 8+k_1 & s+5+k_2 \end{bmatrix}$$

$$\Delta = (s-1)(s+5+k_2) + 8+k_1$$

$$= s^2 + s(4+k_2) + 3 + k_1 - k_2 = 0.$$

$$s^2 + 4s + 11.49 = 0$$

$$\therefore k_2 = 0 \quad \& \quad k_1 = 8.49$$

$$\therefore K = [8.49 \quad 0].$$

$\boxed{9}$

$4 + k_2 = 4$

$k_2 = 0$

$3 + k_1 - k_2 = 11.49$

$3 - 0 + k_1 = 11.49$

$k_1 = 8.49$

c) ~~Desired~~ $\zeta = 0.59$

$$T_s = \frac{4}{\zeta\omega_n} = 1$$

$$\therefore \omega_n = 6.78$$

Desired CE for the observer.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s^2 + 8s + 45.97 = 0.$$

Q4.

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$$A - LC = \begin{bmatrix} 1 & 1 \\ -8 & -5 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-l_1 & 1 \\ -(8+l_2) & -5 \end{bmatrix}$$

$$sI-(A-LC) = \begin{bmatrix} s+l_1-1 & -1 \\ 8+l_2 & s+5 \end{bmatrix}$$

$$\Delta = (s+l_1-1)(s+5) + 8 + l_2 = s^2 + 5s + l_1 s + 5l_1 - s - 5 + 8 + l_2$$

$$= s^2 + s(l_1+4) + 3 + 5l_1 + l_2$$

$$s^2 + 8s + 45.97$$

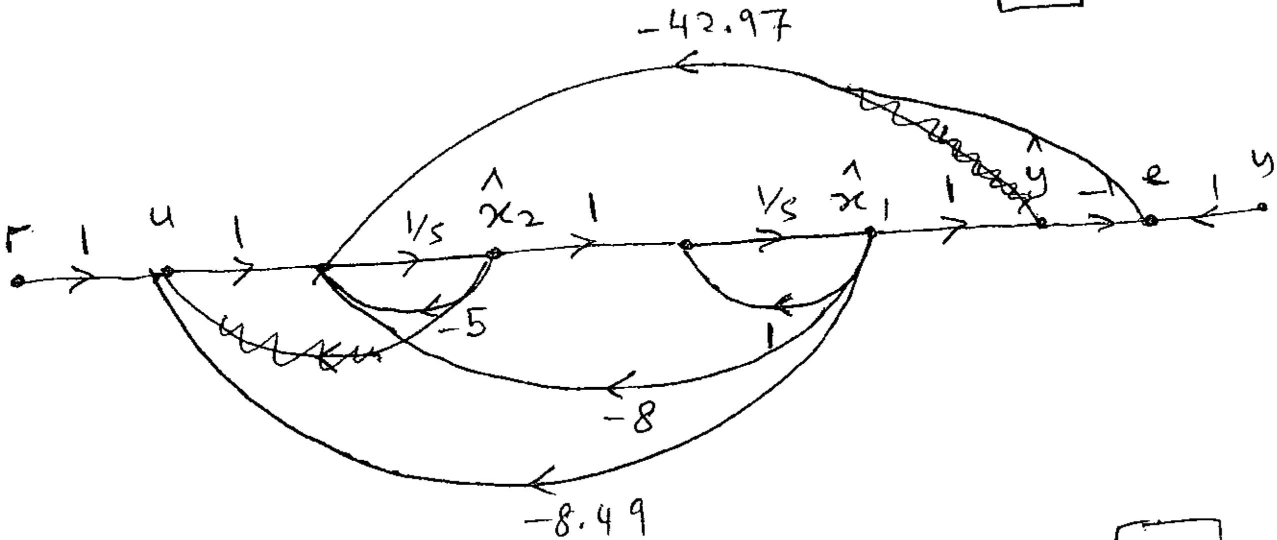
$\therefore\ l_1 = 0$ & $l_2 = 42.97$.

$$\therefore\ L = \begin{bmatrix} 0 \\ 42.97 \end{bmatrix}$$

$l_1 + 4 = 8$

$l_1 = 4$

$\boxed{9}$



$\boxed{3}$

Q5.

$$K = \begin{bmatrix} -1 & -k \end{bmatrix}$$

$$H = A + BK = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & -k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1-k \end{bmatrix}$$

$$J = \int (x^T x + u^T u)\, dt = \int (x^T x + x^T K^T K x)\, dt$$

$$= \int x^T (1 + K^T K) x \, dt$$

$$\therefore Q = I + K^T K = \begin{bmatrix} -1 \\ -k \end{bmatrix} \begin{bmatrix} -1 & -k \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & k \\ k & 1+k^2 \end{bmatrix}$$

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$

Using Lyapunov function:

$$H^T P + PH = -Q.$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 1-k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1-k \end{bmatrix} = -\begin{bmatrix} 2 & k \\ k & 1+k^2 \end{bmatrix}$$

Element (1,1)

$$-2p_{12} = -2$$

$$\therefore p_{12} = 1.$$

Q5

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Element (1, 2)

$$-P_{22} + P_{11} + 1 \cancel{-k} = \cancel{-k}$$

$$\therefore P_{11} = P_{22} - 1 .$$

Element (2, 2)

$$2[1 + (1-k)P_{22}] = -(1+k^2)$$

$$(1-k)P_{22} = -\frac{1+k^2}{2} - 1$$

$$= -\frac{3+k^2}{2}$$

$$\therefore P_{22} = \frac{k^2+3}{2(k-1)}$$

$$\therefore P_{11} = \frac{k^2+3}{2(k-1)} - 1$$

$$= \frac{k^2-2k+5}{2(k-1)}$$

$$\therefore P = \begin{bmatrix} \frac{k^2-2k+5}{2(k-1)} & 1 \\ 1 & \frac{k^2+3}{2(k-1)} \end{bmatrix}$$

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15.

Q5

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b) $J = x_0^T P x_0 - \cancel{x_\infty^T P x_\infty}^{\to 0}$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= p_{22}$$

$$= \frac{k^2+3}{2(k-1)}$$

$$\frac{dJ}{dk} = \frac{2(k^2+3) - 2k(2k-1)}{4(k-1)^2} = 0$$

$2k^2+6-4k^2+2k = 0.$

$k^2 - k - 3 = 0$

$(k-3)(k+1) = 0.$

When $k = 3$, $J = \frac{12}{4} = 3.$

$k = 3.1$, $J = 3.0024$

$\therefore J_{min} = 3$ occurs when $k=3$.

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