System Identification & Parameter Estimation

Wb2301: SIPE lecture 3

Impulse and frequency response functions

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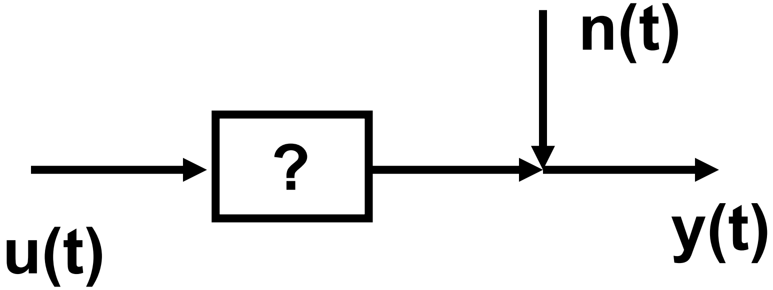
Contents

- Estimation of Impulse response functions (IRF)
- Estimation of Frequency response functions (FRF)
- Improving the estimate of spectral densities
  - ‘Welch’ method
  - Frequency averaging
- Open-loop vs. closed-loop: causality!
- Estimation of linearity: coherence functions

Basic identification with cross-covariance



$$y(t) = n(t) + \int h(t')u(t - t')dt'$$

multiply with $u(t-\tau)$: $u(t-\tau)y(t) = u(t-\tau)n(t) + \int h(t')u(t-\tau)u(t - t')dt'$

$$C_{uy}(\tau) = C_{un}(\tau) + \int h(t')C_{uu}(\tau - t')dt'$$

white noise: $C_{uu}(\tau) = 0 \text{ for } \tau \neq 0; \quad C_{uu}(0) = 1$

$$C_{uy}(\tau) = C_{un}(\tau) + h(\tau)$$

Other 'tricks' needed when u(t) is not white

Impulse response function

- Impulse response function *h(t)*
  - reaction (output) of a system in time after an impulse

- Impulse, or dirac:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

- Output with other input
  - Convolution with *h(t)*

$$y(t) = \int h(\tau)u(t-\tau)d\tau$$

Impulse response function

- Causal system: h(t)=0 for t<0
- Finite memory: h(t)=0 for t>T

$$y(t) = \int_{o}^{T} h(\tau)u(t-\tau)d\tau$$

- Discrete

$$\boxed{y(t) = \sum_{\tau=0}^{T-1} h(\tau)u(t-\tau)\Delta\tau}$$

Estimation of impulse response function (IRF)

- Westwick & Kearney, p. 106-115
  - Direct estimation
  - Least-squares regression
  - Correlation-based methods

Direct Estimation of impulse response function (IRF)

- Apply (multiple) impulses
  - True impulse is physically impossible!
  - Alternative: pulse with fixed width and height

- Disadvantages
  - Impractical
  - Amplitude constraints
  - Noise

IRF via least squares regression

$$y(t) = \sum_{\tau=0}^{T-1} h(\tau)u(t-\tau)\Delta\tau$$

- Rewrite convolution as matrix

$$y = Uh$$

with

$$U = \begin{bmatrix} u(1) & 0 & 0 & \dots & 0 \\ u(2) & u(1) & 0 & \dots & 0 \\ u(3) & u(2) & u(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ u(N) & u(N-1) & u(N-2) & \dots & u(N-T+1) \end{bmatrix}$$

IRF via least squares regression

- z is measurement of y, with noise n(t):

$$z(t) = y(t) + v(t)$$

$$z = Uh + v$$

- Solution via linear least-squares regression (see W&K p.26):

$$\hat{h} = \left(U^T U\right)^{-1} U^T z$$

IRF via correlation-based methods

- See Westwick and Kearney, not discussed

Frequency Domain Expressions

- Discrete Fourier Transform:

$$U(f) = \Im\left(u(t)\right) = \sum_{t=1}^{N} u(t) e^{-j2\pi\frac{ft}{N}}$$

- where *f* takes values *0, 1, ..., N-1* multiples of $\Delta f = \frac{1}{N\Delta t}$

- Inverse Fourier Transform:

$$u(t) = \Im^{-1}\left(U(f)\right) = \frac{1}{N}\sum_{f=1}^{N} U(f) e^{j2\pi\frac{ft}{N}}$$

Frequency domain models

- Time-domain: convolution integral

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$

- 'Convolution in time-domain is multiplication in frequency domain' (and vice versa)

frequency domain: $$Y(f) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \right) e^{-2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} u(t-\tau) e^{-2\pi f t} dt d\tau$$

$$= U(f) \int_{-\infty}^{\infty} h(\tau) e^{-2\pi f \tau} d\tau$$

$$= U(f) H(f)$$

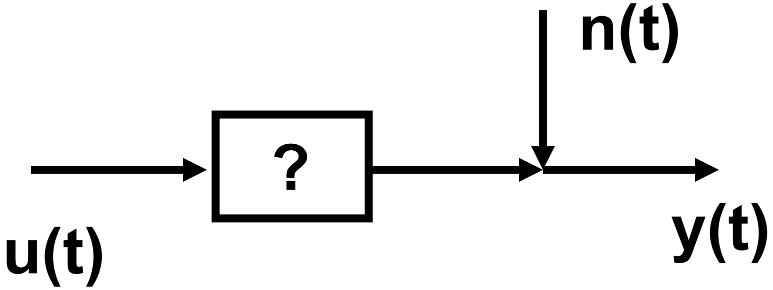
Estimation of frequency response function (FRF)

- Sinusoidal frequency response testing
  - Apply single sinusoid

- Stochastic frequency response testing
  - Noise based

- Periodic frequency response testing
  - Pintelon & Schoukens, discussed in Lecture 4

Basic identification with spectral densities



$$y(t) = n(t) + \int h(t')u(t - t')\,dt'$$

multiply with $u(t-\tau)$: $u(t-\tau)\,y(t) = u(t-\tau)\,n(t) + \int h(t')u(t-\tau)u(t-t')\,dt'$

$$C_{uy}(\tau) = C_{un}(\tau) + \int h(t')C_{uu}(\tau - t')\,dt'$$

Fourier transform: $S_{uy}(f) = S_{un}(f) + H(f)S_{uu}(f)$

if $S_{un}(f) = 0$: $H(f) = \dfrac{S_{uy}(f)}{S_{uu}(f)}$

Spectral densities (lecture 2)

- **Spectral density**
  - Defined as Fourier transform of cross-correlation (indirect approach)

$$\hat{S}_{uy}(f) = \sum_{\tau=0}^{N-1} \hat{\Phi}_{uy}(\tau) e^{-j2\pi\frac{f\tau}{N}}$$

  - Only evaluated for till $f_s/2$; $S_{uy}(-f) = S_{uy}(f)^*$
  - Direct approach via transformed signals

$$\hat{S}_{uy}(f) = \frac{1}{N} U^*(n\Delta f) Y(n\Delta f)$$

- **Properties auto-spectral density**
  - Real valued (no imaginary part)
  - Parceval: area under S_{uu} is related to signal's variance
- **Properties cross-spectral density**
  - Complex values
  - Gives interdependency between two signals (gain&phase)

Properties of the spectral density estimator

- Matlab demo: Lec3_length_Suu.m
  Relation between *number of samples* and *variance of the estimator*

- Increasing the number of samples:
  - Longer observation: increased frequency resolution
  - Higher sample frequency: increased frequency bandwidth
  - => Variance of the (raw) estimator remains equal !

=> The raw estimator is **not consistent**

- The raw estimate for the auto-spectral density is sometimes called the **periodogram**

Improving the estimate

- Common techniques to improve the spectral estimate
  - Frequency averaging
  - Welch method

- Other (=old) methods:
  - Multiply cross-covariance with window before DFT
  - Convolve the spectral density with window
  - Will be discussed during Lecture 4

- Not to confuse with signal windowing (also ‘tapering’); is done before DFT, will be discussed during Lecture 4

Improving the estimate

- Reduce variance of spectral estimator by averaging
  - either over multiple repetitions,
  - or over adjacent frequencies

- Basic ‘idea’:
  - Multiple repetitions: each realizations has the same frequency content.
  - Frequency averaging: the spectral density is often smooth, i.e. adjacent frequencies contain (approx.) the same information.

- => Averaging will reduce the effect of noise, as the noise has zero mean.

Welch method (averaged periodogram)

- Divide data in multiple segments
- Calculate spectral density for each segment
- Average over the segments

$$\hat{S}_{uu}(f) = \frac{1}{D}\sum_{d=1}^{D} S_{uu}(f) = \frac{1}{DN_D}\sum_{d=1}^{D} U(-f)U(f)$$

Frequency averaging

- Calculate the raw spectral density
- Average over adjacent frequencies

$$\hat{S}_{uu}(f_c) = \frac{1}{D}\sum_{d=1}^{D}\hat{S}_{uu}(f_d); \quad f_c = \frac{1}{D}\sum_{d=1}^{D} f_d$$

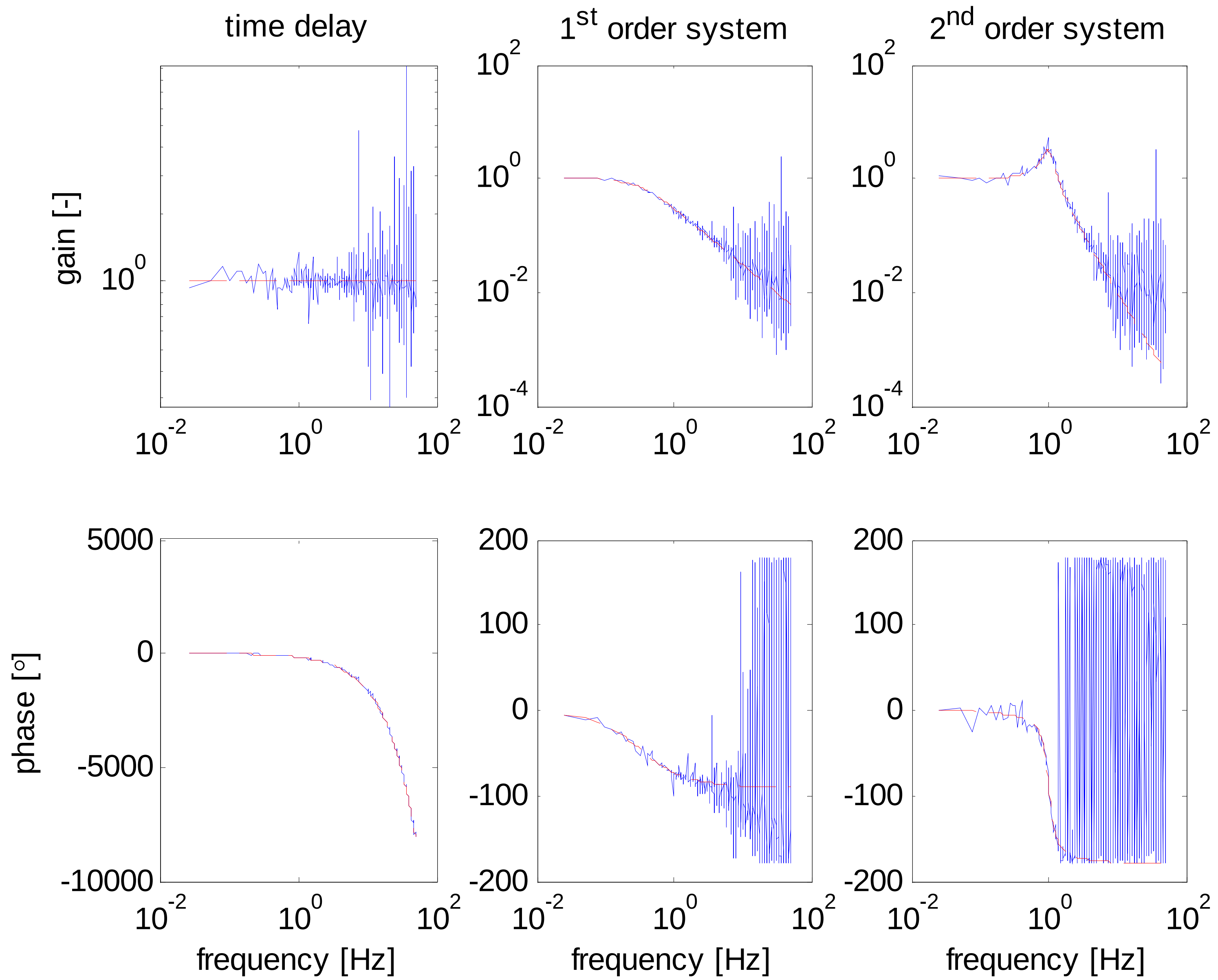
- Possible drawback: can introduce bias when the power at adjacent frequencies is not similar (e.g. sharp oscillatory peaks)

=> Matlab demo: Lec3_SmoothSuu.m

Frequency response of some basic systems

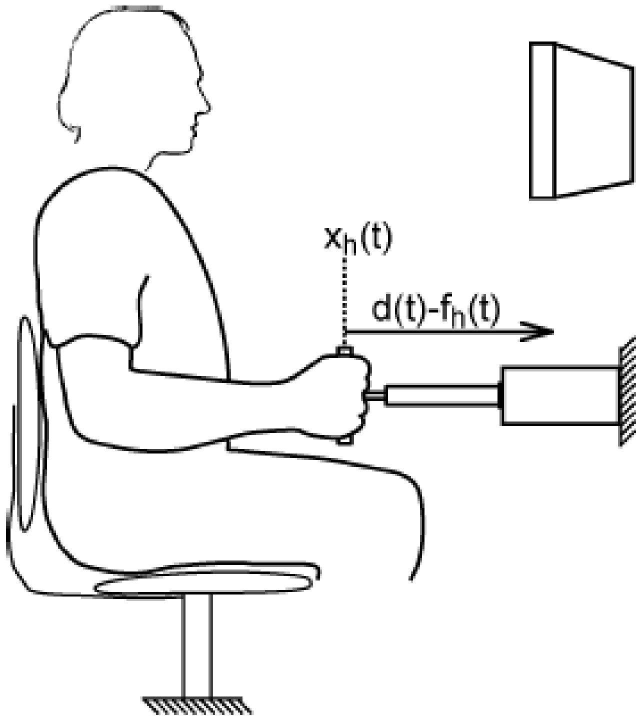
- Example frequency response functions of different systems (Matlab demo: Lec2_example_systems.m):
  - Time delay
  - 1st order
  - 2nd order

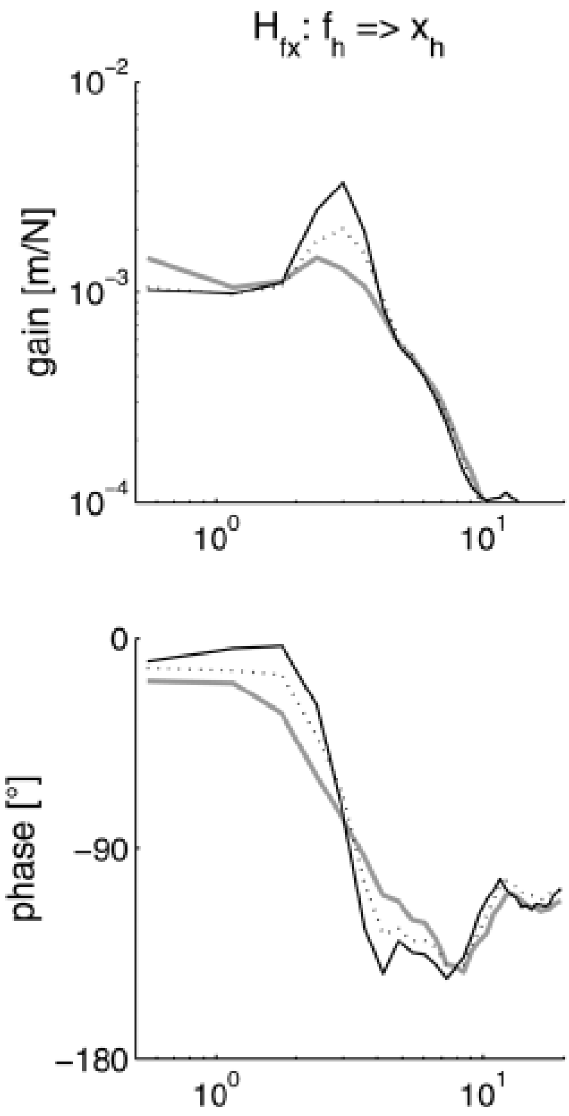
- Frequency response functions (FRF) of different systems:
  - Conclusion: the system can ‘easy’ be recognized from a frequency response functions (system order, natural frequency, relative damping, etc)


time delay
1st order system
2nd order system
gain [-]
phase [°]
10^2
10^0
10^-2
10^-4
5000
0
-5000
-10000
200
100
-100
-200
frequency [Hz]
frequency [Hz]
frequency [Hz]

Example FRF

- Admittance of human arm





Coherence and coherency

- Coherency

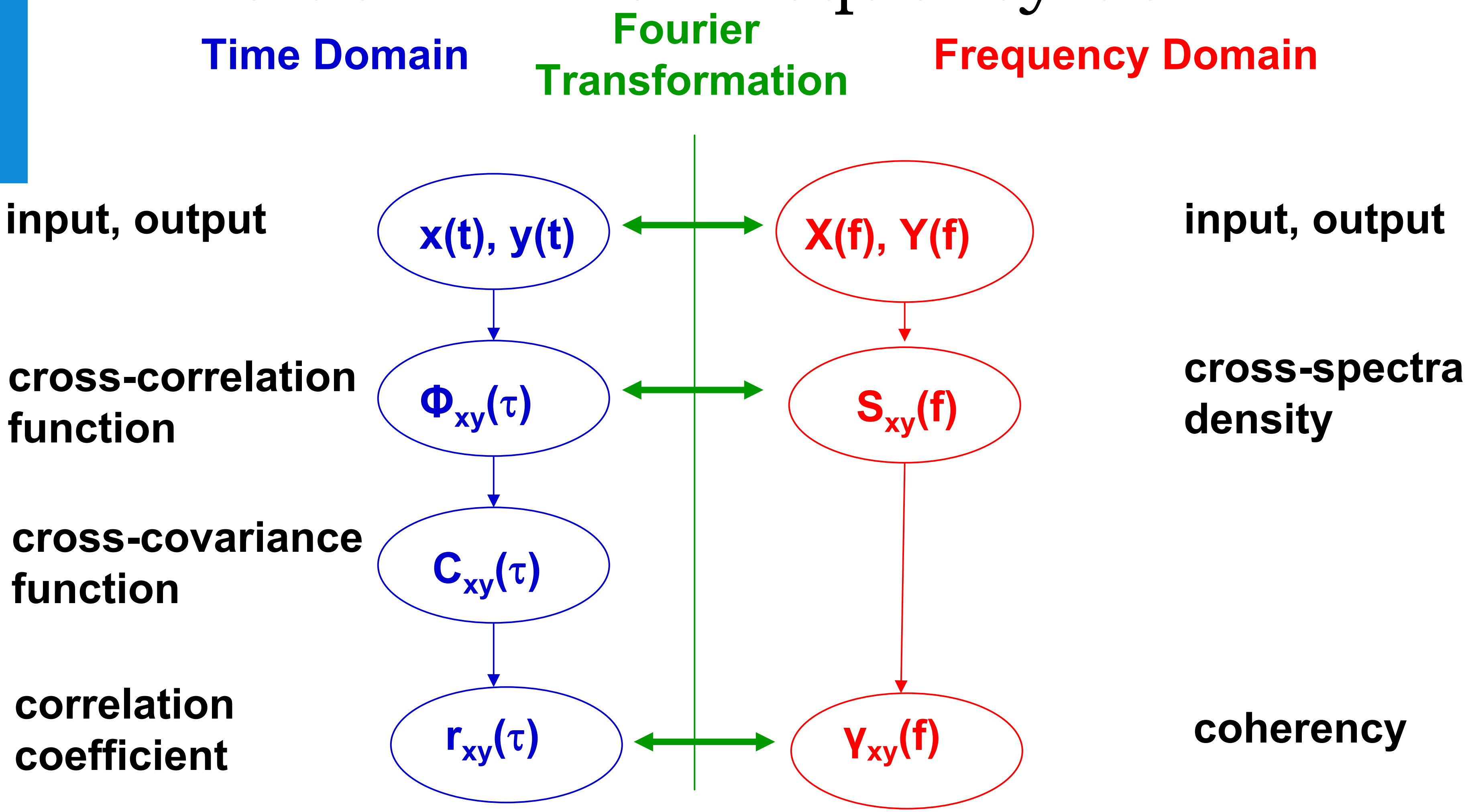
$$\gamma_{uy}(f) = \frac{S_{uy}(f)}{\sqrt{S_{yy}(f)S_{uu}(f)}}$$

- Coherence

$$\gamma_{uy}{}^{2}(f) = \frac{\left|S_{uy}(f)\right|^{2}}{S_{yy}(f)S_{uu}(f)}$$

- Coherence
  - Real valued, between 0 and 1
  - Squared coherency
- Coherency has a phase, which represent the relative delay between the signals

Time-domain vs. Frequency-domain

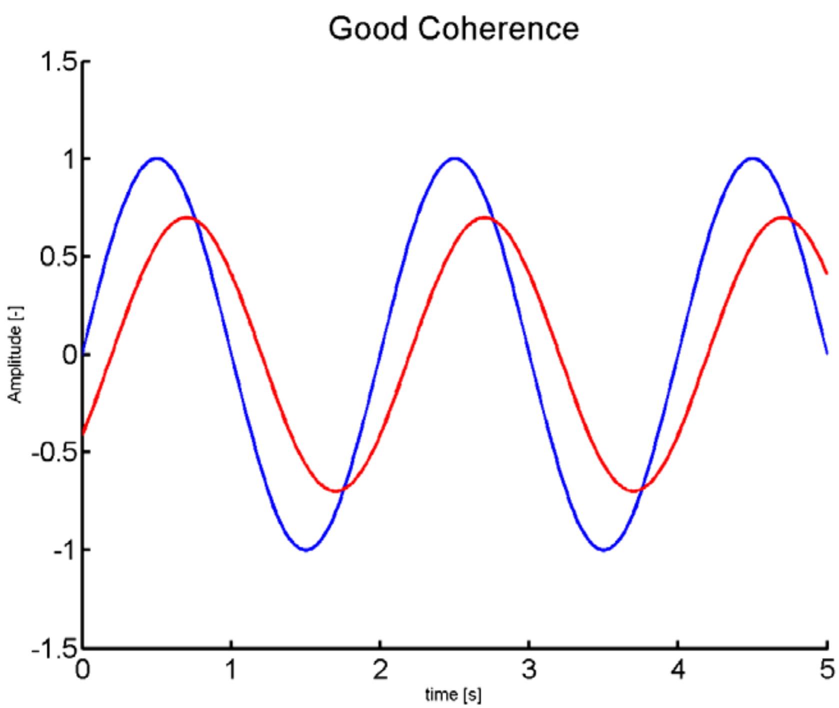


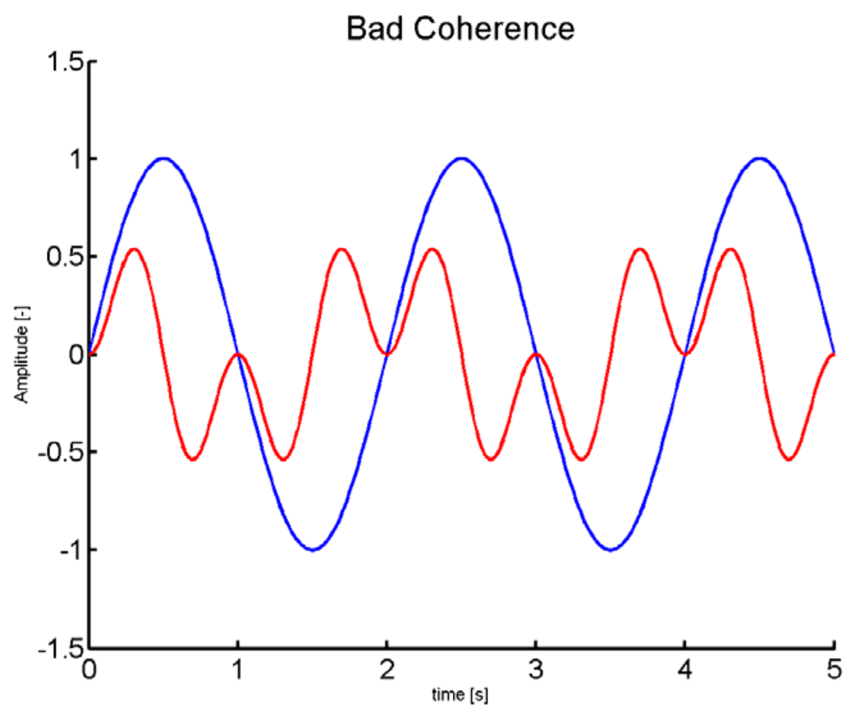
Coherence γ_{uy}^2

Coherence:

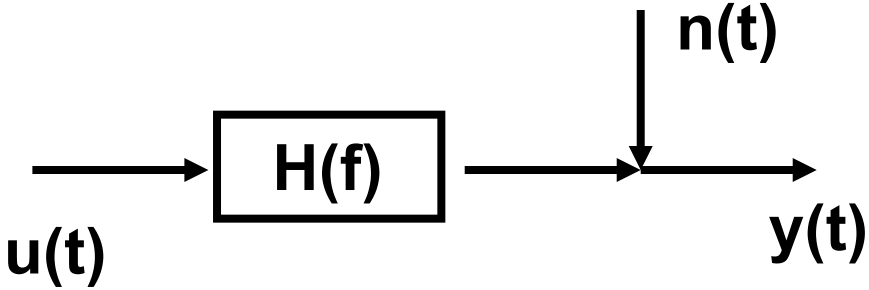
$$\gamma_{uy}^{2}(f) = \frac{\left|S_{uy}(f)\right|^{2}}{S_{yy}(f)S_{uu}(f)} \qquad 0 \le \gamma_{yu}^{2}(f) \le 1$$

Coherence: Linear relationship between input and output, irrespective of the type of system in between.





Theoretical open loop coherence



$$S_{uy}(f) = H(f)S_{uu}(f) + S_{un}(f)$$

$$\text{if} \quad S_{un}(f) = 0; \quad S_{yy}(f) = \left|H(f)\right|^2 S_{uu}(f) + S_{nn}(f)$$

$$\gamma_{uy}{}^2(f) = \frac{\left|S_{uy}(f)\right|^2}{S_{yy}(f)S_{uu}(f)} = \frac{\left|H(f)S_{uu}(f)\right|^2}{S_{uu}(f)\left(\left|H(f)\right|^2 S_{uu}(f) + S_{nn}(f)\right)}$$

$$\gamma_{uy}{}^2(f) = \frac{1}{1 + \dfrac{S_{nn}(f)}{\left|H(f)\right|^2 S_{uu}(f)}}$$

Coherence $\gamma_{uy}{}^2$

$$\hat{\gamma}_{uy}{}^2(f) = \frac{\left|\hat{S}_{uy}(f)\right|^2}{\hat{S}_{yy}(f)\hat{S}_{uu}(f)} = \frac{\left|\frac{1}{N}U(-f)Y(f)\right|^2}{\frac{1}{N}Y(-f)Y(f)\cdot\frac{1}{N}U(-f)U(f)}$$

$$\hat{\gamma}_{uy}{}^2(f) = \frac{\left|U(-f)Y(f)\right|^2}{\left|Y(f)\right|^2\cdot\left|U(f)\right|^2} = 1$$

- With the ‘raw’ spectra the coherence equals 1 !
- Smoothing is required. However as a result of the squared cross-spectrum the coherence will be overestimated (bias)

Coherence

- Coherence indicates if two signals are linearly related
  - Reduced by additional signals (noise) and nonlinearities
- Raw estimate of coherence is always 1
  - Artifact!
- Smoothing of spectral densities is required to get a realistic estimate
- Effect of smoothing on estimator for the coherence
  - Coherence estimator is always biased; overestimated as a result of the square in the coherence (structural error => bias)
  - With averaging the estimator approaches the 'true' value

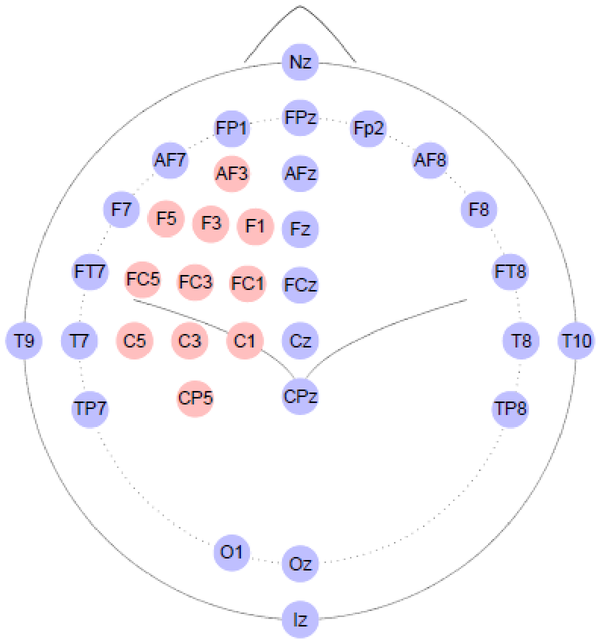
Examples from neuroscience

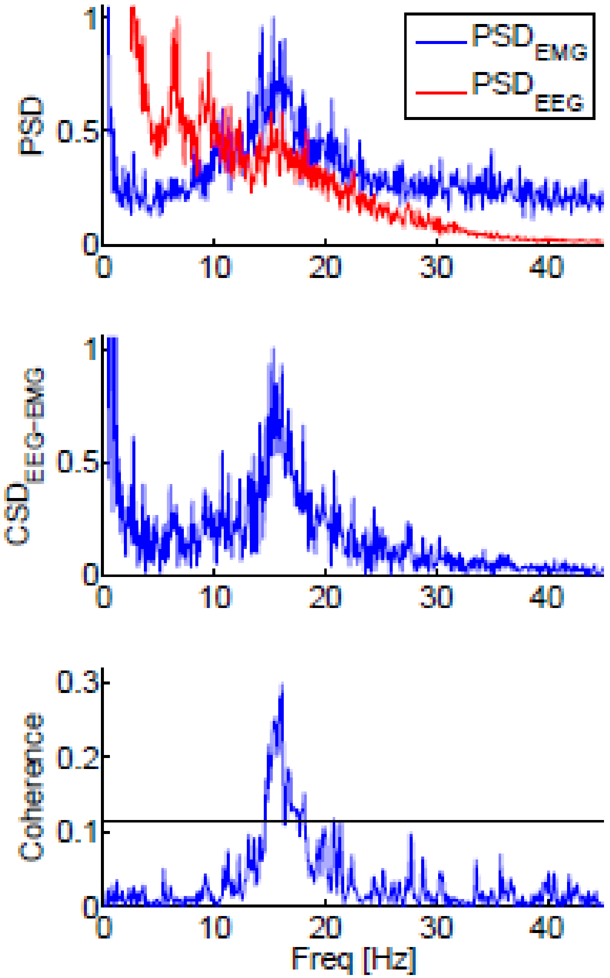
- Corticomuscular coherence (CMC)
  - Calculated between EEG and EMG. It is thought to represent a functional connection between brain (motor cortex) and muscles. Best found during isometric contractions in subjects.

- Intermuscular coherence
  - Calculated between EMGs of different muscles. It indicates that muscles are driven by one common ‘drive’. Found in specific motor disorders (o.a. myoclonus dystonia). Normally each muscle is activated by a specific area in the motor cortex, as such no significant coherence exists.

Corticomuscular coherence (CMC)

- Example CMC





Significance of coherence

- With corticomuscular or intermuscular coherence, one is interested if coherence exists, i.e. is it significantly different from zero coherence (no coherence)

- If coherence is higher than significance level, signals are linearly related (often very weak, as coherence is between 0.1-0.3)

- Note some authors present coherency (and other do not even explicitly mention what is presented)
  - If coherence = 0.1 - 0.3, than coherency = 0.32 – 0.55)

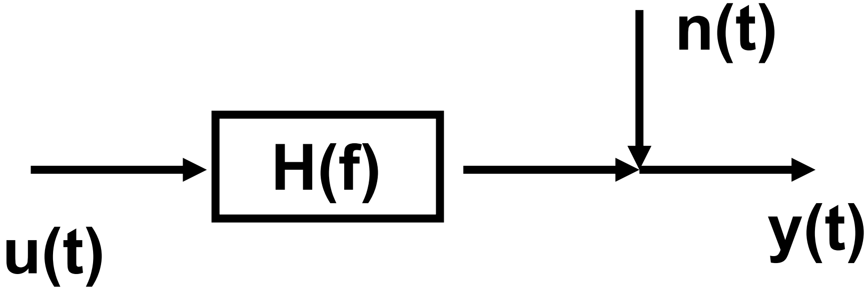
Summary: effect of Welch method or frequency averaging

- Estimators for the spectral densities
  - Variance decreases with averaging
  - Resolution decreases with averaging!

- Effect on estimator of FRF
  - Variance decreases with averaging
  - Able to estimate at higher frequencies (where normally the noise would deteriorate the estimate)

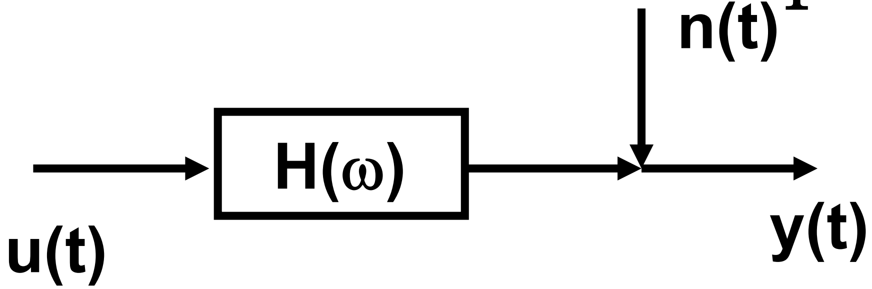
- Effect on estimator for the coherence
  - Coherence estimator is always biased
  - With averaging the estimator approaches the ‘true’ value

Identification of open loop systems



- **Open loop identification in frequency domain**
  - In most cases the input and the noise are not correlated
    ($S_{nu}(f)=0$ for all f)
  - Estimator: $\hat{H}(f) = \dfrac{\hat{S}_{uy}(f)}{\hat{S}_{uu}(f)}$ (^ denotes estimate)

Identification of open loop system II



$Y(\omega) = H(\omega).U(\omega) + N(\omega)$

Additional signal $Z(\omega)$:

$Z(-\omega).Y(\omega) = Z(-\omega).H(\omega).U(\omega) + Z(-\omega).N(\omega)$

$S_{zy}(\omega) = H(\omega).S_{zu}(\omega) + S_{zn}(\omega)$

if $Z(\omega)$ is uncorrelated with $N(\omega)$:

$S_{zn}(\omega) = 0$

$S_{zy}(\omega) = H(\omega).S_{zu}(\omega)$

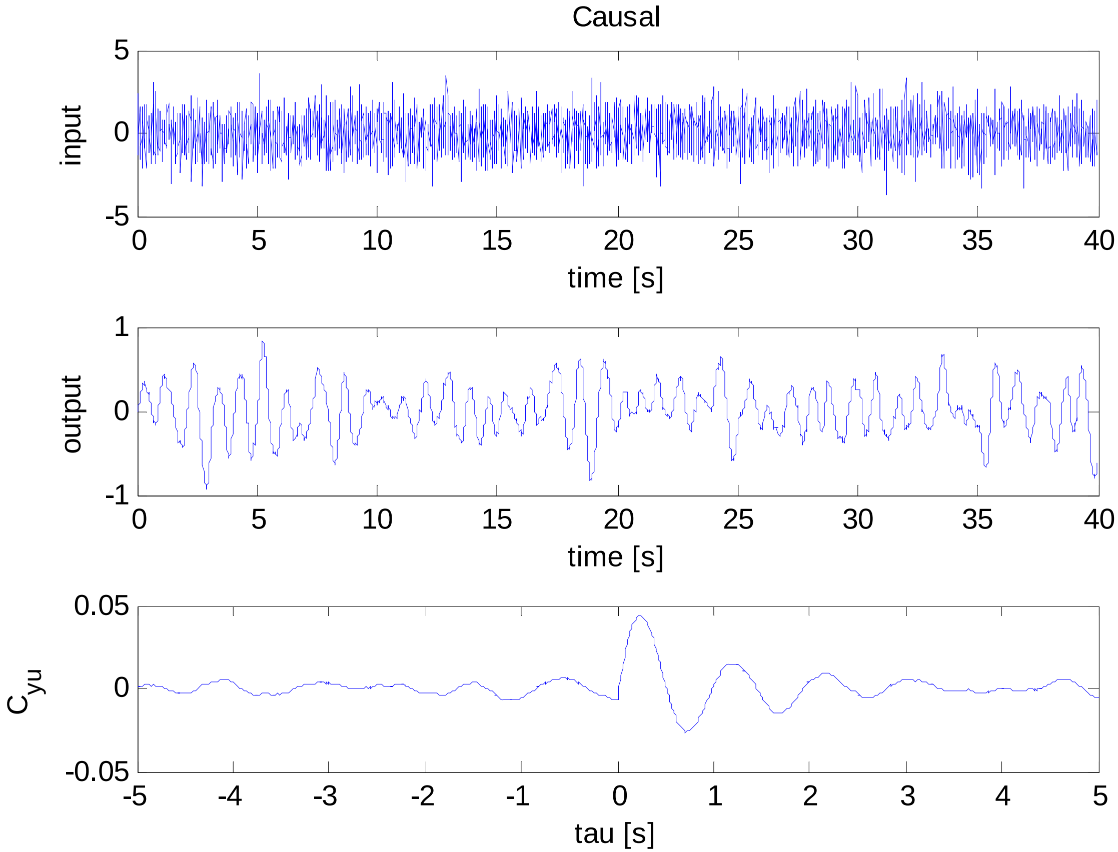
if $U(\omega)$ is uncorrelated with $N(\omega)$, then $Z(\omega) = U(\omega)$:

$S_{uy}\omega) = H(\omega).S_{uu}(\omega)$

$H(\omega) = S_{uy}(\omega)/S_{uu}(\omega)$

Causality and cross-covariance

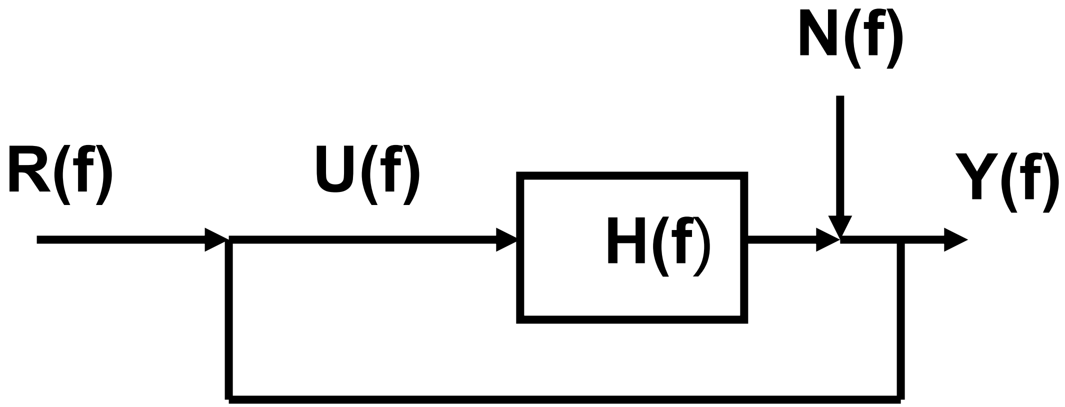
- Demo in Matlab: Lec3_example_causelity_Cyu.m



Basic theory: causality

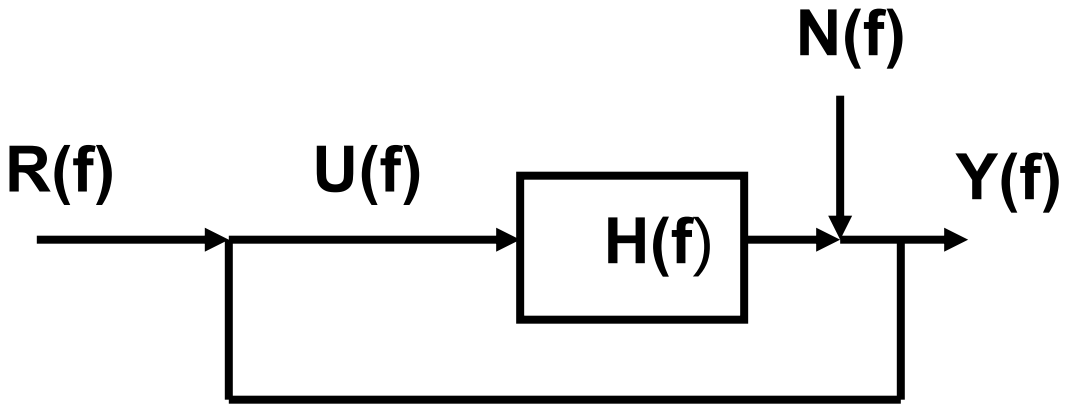
- Physical systems are causal: output depends on *previous* values of the input.
- Anti-causal: depends on *only future* values of the input.
  => input-output are exchanged
- Non-causal: depends on previous and future values of the input.
  => closed-loop systems
  => two parallel subsystems

Identification of a system in closed loop



- Closed loop: $S_{un}(f) \neq 0$!
- Consequently: $H(f) \neq S_{uy}(f)/S_{uu}(f)$

Identification of a system in closed loop

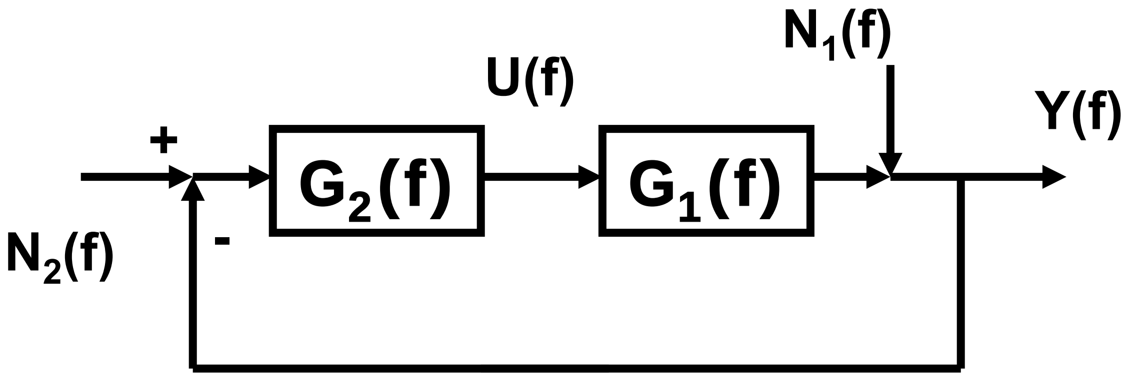


- Closed loop: $S_{un}(f) \neq 0$!
- Consequently: $H(f) \neq S_{uy}(f)/S_{uu}(f)$

Examples of closed loop identification

- Human-machine interaction (driving, steering, etc.)
  - Interaction between two systems!
- Human motion control
  - Muscle force depends on activation, activation depends on reflexes, reflexes depend on movement.
- Chemical/nuclear plants
  - Plant is unstable, so a controller is needed.
  - Identification around a desired operation point, a controller is required to keep the system in the desired operation point.

Identification in the closed loop



- What happens if we would use an open loop estimator for a system in closed loop:

$$G_1'(f) = \frac{S_{uy}(f)}{S_{uu}(f)}$$

- What is relation between G_1` and true G_1

Readings

- Book Westwick & Kearney
  - Chapter 1, all (lecture 1)
  - Chapter 2, sec. 2.1 – 2.3.4 (lecture 1+2)
  - Chapter 3, sec. 3.1 – 3.2 (lecture 2)
  - Chapter 5, sec. 5.1 – 5.3 (lecture 3)

- Book Pintelon & Schoukens
  - Chapter 1, sec 1.1 – 1.4 (optional, lecture 1)
  - Chapter 2, all (lecture 4)
  - Chapter 4, all (lecture 4)

- Articles
  - de Vlugt et al. (lecture 5)