System Identification & Parameter Estimation

Wb2301: SIPE lecture 2

Correlation functions in time & frequency domain

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- Estimators for correlation functions
- Fourier transform
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  - Cross-spectral density
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Lecture 1: (auto)correlation functions

- Autocorrelation function

$$\Phi_{xx}(\tau) = E\left[x(t-\tau)x(t)\right]$$

- Autocovariance function

$$C_{xx}(\tau) = E\left[\left(x(t-\tau)-\mu_x\right)\left(x(t)-\mu_x\right)\right] = \Phi_{xx}(\tau)-\mu_x^2$$

- Autocorrelation coefficient

$$r_{xx}(\tau) = E\left[\left(\frac{x(t-\tau)-\mu_x}{\sigma_x}\right)\left(\frac{x(t)-\mu_x}{\sigma_x}\right)\right] = \frac{\Phi_{xx}(\tau)-\mu_x^2}{\sigma_x^2} = \frac{C_{xx}(\tau)}{C_{xx}(0)}$$

Lecture 1: autocovariance function

- Autocovariance function:

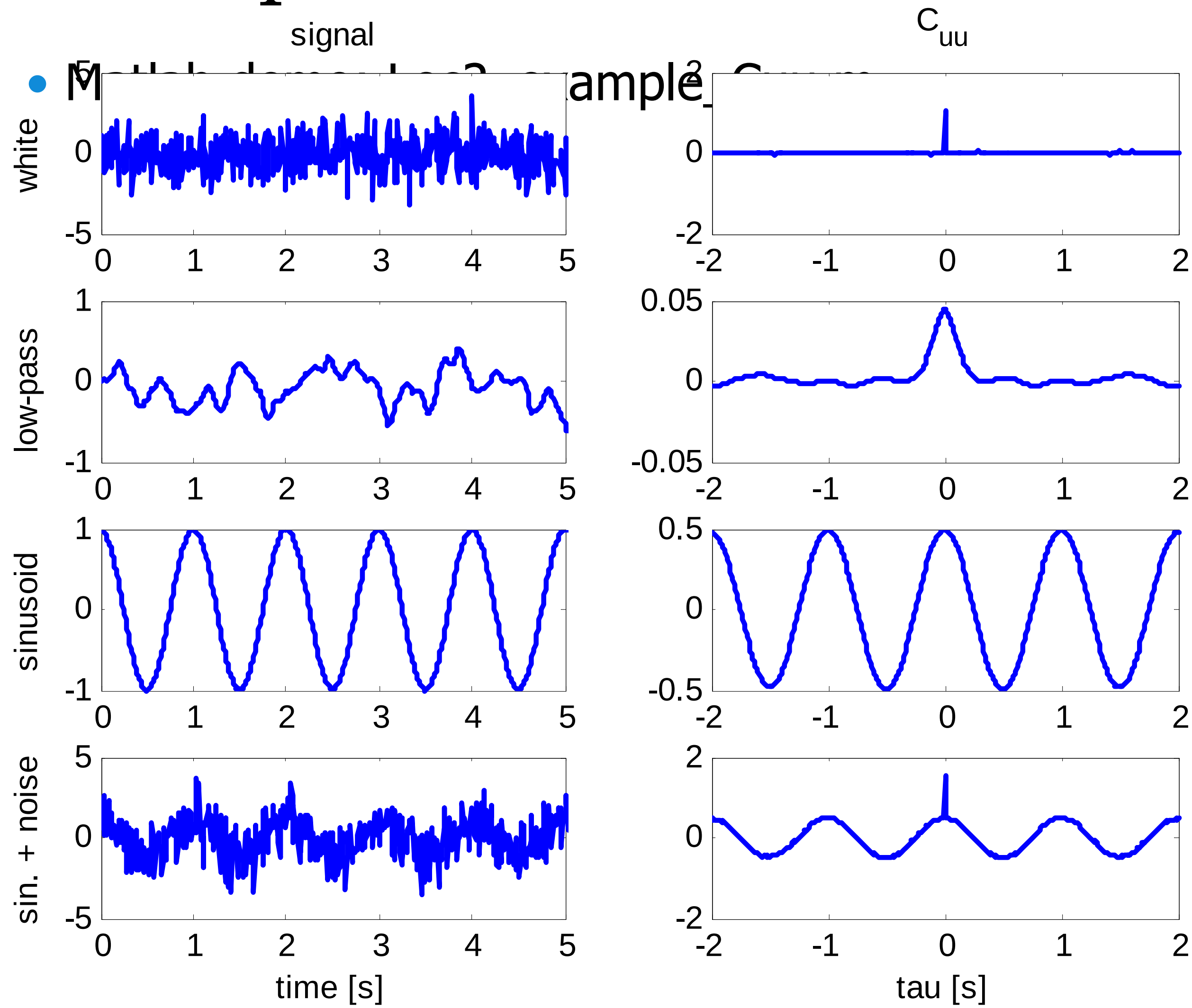
$$C_{xx}(\tau) = E\left[\left(x(t-\tau)-\mu_x\right)\left(x(t)-\mu_x\right)\right]$$
$$= E\left[x(t-\tau)x(t)\right] - E\left[x(t-\tau)\mu_x\right] - E\left[\mu_x x(t)\right] + E\left[\mu_x^2\right]$$
$$= \Phi_{xx}(\tau) - \mu_x^2 - \mu_x^2 + \mu_x^2$$
$$= \Phi_{xx}(\tau) - \mu_x^2$$

- If $\mu = 0$, then autocovariance and autocorrelation functions are identical

- At zero lag: $C_{xx}(0) = E\left[\left(x(t)-\mu_x\right)^2\right] = \sigma_x^2$

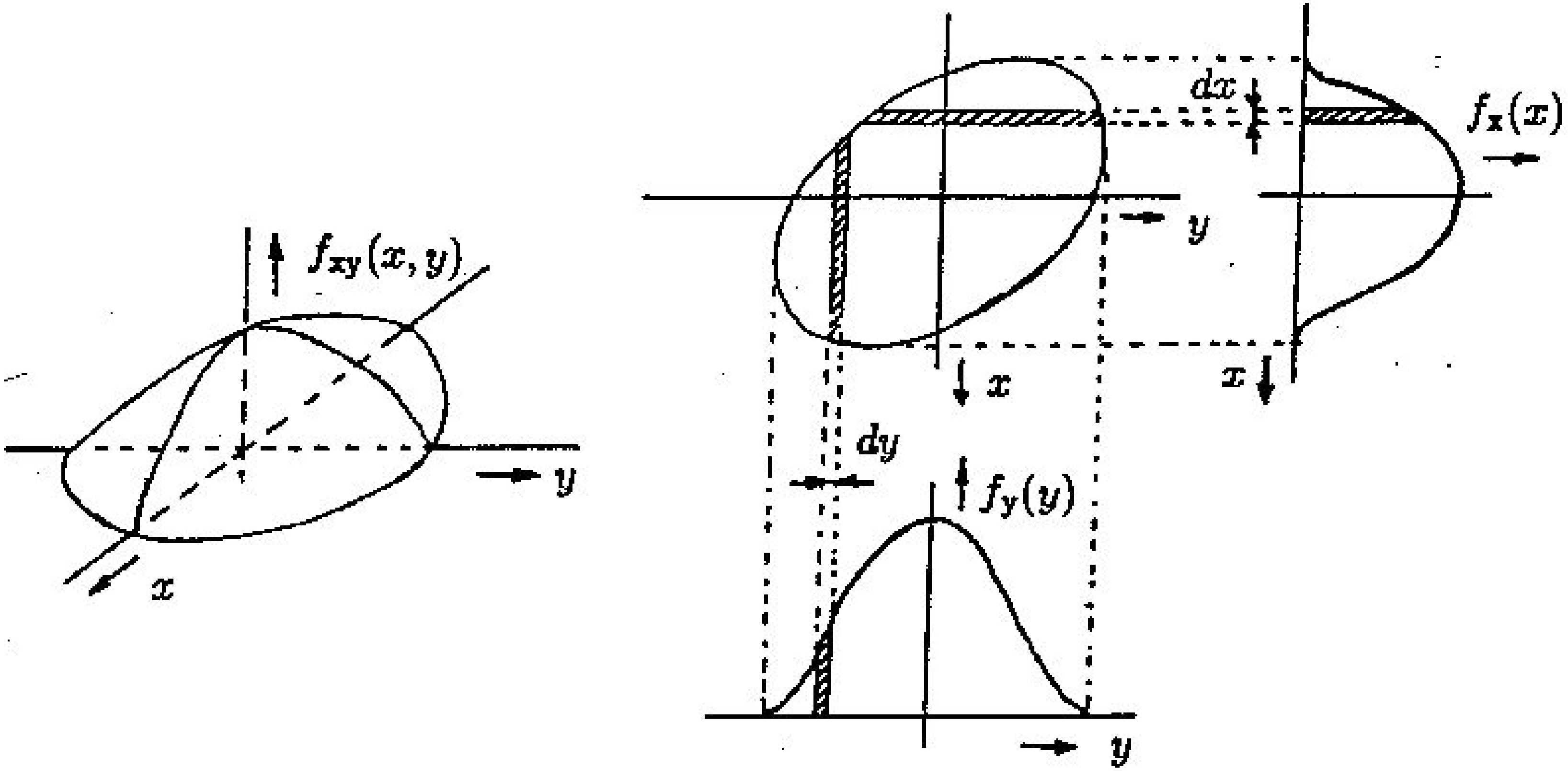
Example autocovariances

- M ample_



2D Probability density function

$$f_{\overline{xy}}(x, y; \tau)dxdy = \Pr\{x < \overline{x}(t;\zeta) \leq x + dx \cap y < \overline{y}(t+\tau;\zeta) \leq y + dy\}$$



2D Probability density function

- Probability for certain values of y(t) given certain values of x(t)

- Co-variance of y(t) with x(t): y(t) is related with x(t)
  - No co-variance between y(t) and x(t): 2D probability density function is circular
  - Covariance between y(t) and x(t): 2D probability density function is ellipsoidal

- No co-variance between y(t) and x(t):
  - y(t) and x(t) are independent
  - no relation exist
  - transfer function is zero !

Cross-correlation functions

- Measure for common structure in two signals:
  - Cross-correlation

$$\Phi_{xy}(\tau) = E\left[x(t-\tau)y(t)\right]$$

  - Cross-covariance

$$C_{xy}(\tau) = E\left[\left(x(t-\tau)-\mu_x\right)\left(y(t)-\mu_y\right)\right] = \Phi_{xy}(\tau) - \mu_x\mu_y$$

  - Cross-correlation coefficient

$$r_{xy}(\tau) = E\left[\left(\frac{x(t-\tau)-\mu_x}{\sigma_x}\right)\left(\frac{y(t)-\mu_y}{\sigma_y}\right)\right] = \frac{C_{xy}(\tau)}{\sqrt{C_{xx}(0)C_{yy}(0)}}$$

Example: Estimation of a Time Delay

- Consider the system:

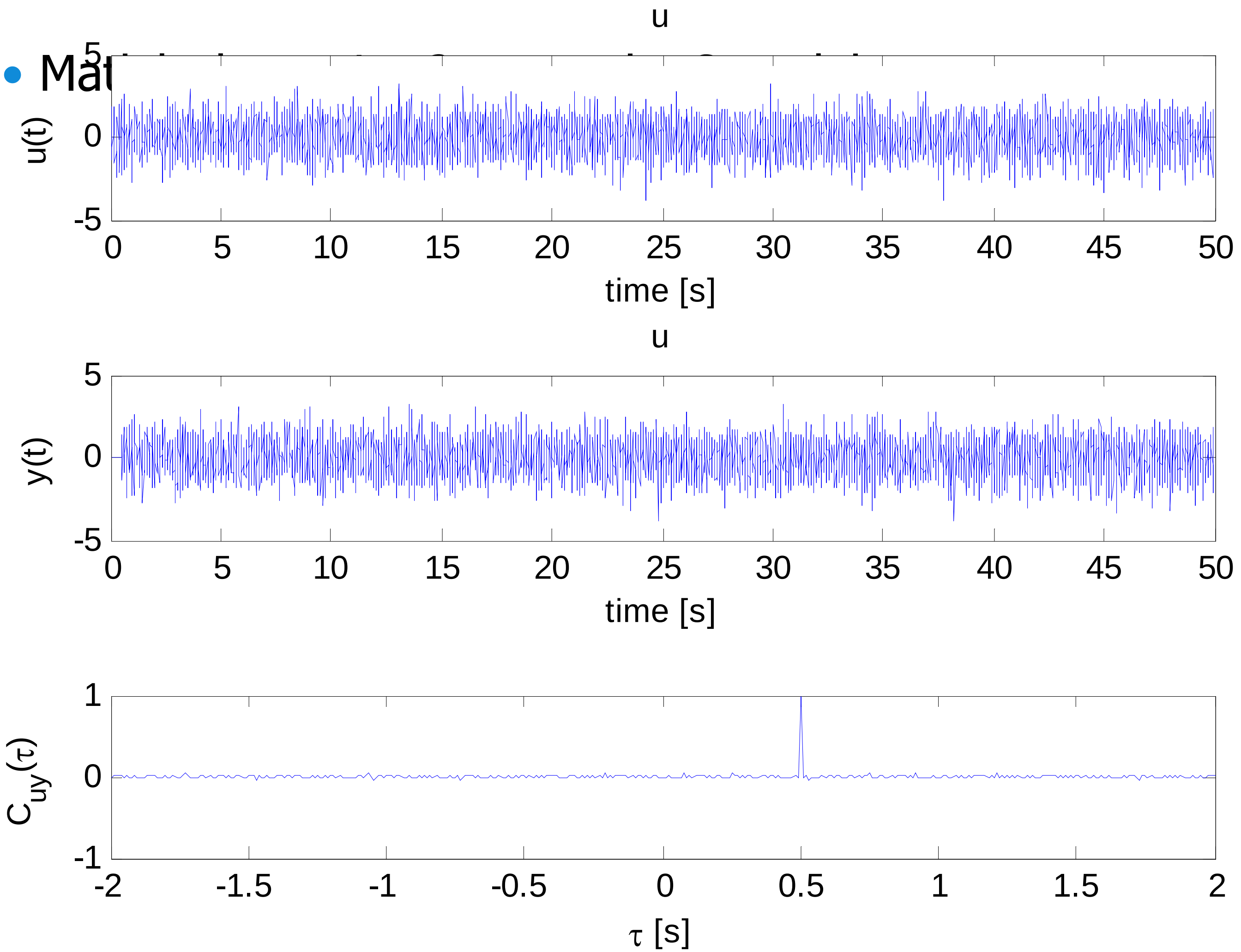
$$y(t) = \alpha x(t - \tau_0) + v(t)$$

  - Additive noise v(t) is stochastic and uncorrelated to x(t),

- Cross-covariance of the measured input and output in case:
  - input x(t) is white noise (stochastic)

Example: Estimation of a Time Delay

- Mat



Effects of Noise on Autocorrelations

- Often, correlation functions must be estimated from measurements.
- Consider $x(t)$ and $y(t)$, corrupted with noise $n(t)$ and $v(t)$ resp.:

$$w(t) = x(t) + n(t)$$
$$z(t) = y(t) + v(t)$$

- Noises have zero mean and are independent of the signals and of each other.
- Autocorrelation: $\Phi_{zz}(\tau) = E\left[\left(y(t-\tau)+v(t-\tau)\right)\left(y(t)+v(t)\right)\right]$

$$= \Phi_{yy}(\tau) + \Phi_{yv}(\tau) + \Phi_{vy}(\tau) + \Phi_{vv}(\tau)$$
$$= \Phi_{yy}(\tau) + \Phi_{vv}(\tau)$$

- Additive noise will **bias** autocorrelation functions!

Effects of Noise on Cross-correlations

- Cross-correlation: $\Phi_{wz}(\tau) = E\left[\left(x(t-\tau)+n(t-\tau)\right)\left(y(t)+v(t)\right)\right]$

$$= \Phi_{xy}(\tau) + \Phi_{xv}(\tau) + \Phi_{ny}(\tau) + \Phi_{nv}(\tau)$$

$$= \Phi_{xy}(\tau)$$

- Additive noise will **not bias** cross-correlation functions!

Noise Reduction

- Longer recordings
  - More ‘information’, same amount of noise

- Repeat the experiment
  - Noise cancels out by averaging from exactly the same inputs

- Improve Signal-to-Noise Ratio
  - Concentrate power at specified frequencies, assuming the noise power remains the same

Properties of Estimators

- Formula given (autocovariances / spectral densities) are **estimators** for the ‘true’ relations.
- What are the properties of these estimators?
  - bias / variance / consistency
- Bias: structural error
- Variance: random error
- Consistent: A consistent estimator is an estimator that converges, in probability, to the quantity being estimated as the sample size grows.

Example: estimator for mean value and variance of a signal

- Signal x_k , with $k=1...N$
- Estimator for the signal mean: $$\hat{\mu}_x = \frac{1}{N}\sum_{k=1}^{N} x_k$$
- Estimator for the signal variance: $$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{k=1}^{N}\left(x_k - \hat{\mu}_x\right)^2$$
- What are the expected values of both estimators? Expectation operator: $E\{.\}$

Estimator for the mean value

$$\hat{\mu}_x = \frac{1}{N}\sum_{k=1}^{N} x_k$$

$$E\{\hat{\mu}_x\} = E\left\{\frac{1}{N}\sum_{k=1}^{N} x_k\right\} = \frac{1}{N}\sum_{k=1}^{N} E\{x_k\}$$

$$E\{x_k\} = E\{x\} = \mu_x$$

$$E\{\hat{\mu}_x\} = \frac{1}{N}\sum_{k=1}^{N} \mu_x = \mu_x$$

Estimator for the variance

$$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{\mu}_x)^2 = \frac{1}{N}\sum_{k=1}^{N}\left(x_k - \frac{1}{N}\sum_{l=1}^{N} x_l\right)^2$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left(x_k{}^2 - \frac{2}{N}\sum_{l=1}^{N} x_k x_l + \frac{1}{N^2}\sum_{l=1}^{N}\sum_{k=1}^{N} x_l x_k\right)$$

$$E\left\{\hat{\sigma}_x^2\right\} = \frac{1}{N}\sum_{k=1}^{N}\left(E\left\{x_k{}^2\right\} - \frac{2}{N}\sum_{l=1}^{N} E\left\{x_k x_l\right\} + \frac{1}{N^2}\sum_{l=1}^{N}\sum_{k=1}^{N} E\left\{x_l x_k\right\}\right)$$

$$E\left\{\hat{\sigma}_x^2\right\} = \frac{1}{N}\sum_{k=1}^{N}\left[\sigma_x^2 + \mu_x^2 - \frac{2}{N}\sigma_x^2\sum_{l=1}^{N} K_{x_k x_l} - 2\mu_x^2 + \frac{1}{N^2}\sigma_x^2\sum_{l=1}^{N}\sum_{k=1}^{N} K_{x_k x_l} + \mu_x^2\right]$$

$$E\left\{\hat{\sigma}_x^2\right\} = \sigma_x^2\left[1 - \frac{2}{N}1 + \frac{1}{N^2}N\right] = \sigma_x^2\left(1 - \frac{1}{N}\right) = \left(\frac{N-1}{N}\right)\sigma_x^2$$

- Estimator is biased and consistent

Estimator for the variance

- 'biased' estimator:

$$\hat{\sigma}_x^2 = \frac{1}{N}\sum_{k=1}^{N}\left(x_k - \hat{\mu}_x\right)^2$$

$$E\left\{\hat{\sigma}_x^2\right\} = \sigma_x^2\left(1 - \frac{1}{N}\right) = \left(\frac{N-1}{N}\right)\sigma_x^2$$

- 'unbiased' estimator: (default!)

$$\hat{\sigma}_x^2 = \frac{1}{N-1}\sum_{k=1}^{N}\left(x_k - \hat{\mu}_x\right)^2$$

$$E\left\{\hat{\sigma}_x^2\right\} = \sigma_x^2$$

Estimates of Correlation Functions

- Cross-correlation: $\Phi_{xy}(\tau) = E\left[x(t-\tau)y(t)\right]$

- Let $x(t)$ and $y(t)$ are **finite time** realizations of ergodic processes. Then:

$$\hat{\Phi}_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-\tau)y(t)dt$$

- Practically, signals are **finite time** and **sampled** every Δt. Signal x(t) is sampled at t=0, Δt, …, (N-1)Δt
  giving x(i) with i=1, 2, …, N.

- Then: $\hat{\Phi}_{xy}(\tau) = \frac{1}{N-\tau} \sum_{i=\tau}^{N} x(i-\tau)y(i)$

  with i = *1, 2, …, N*

Estimates of Correlation Functions

- Unbiased estimator: $$\hat{\Phi}_{xy}(\tau) = \frac{1}{N-\tau}\sum_{i=\tau}^{N} x(i-\tau)y(i)$$
- Variance of the estimator increases with lag!
  To avoid this, divide by N (biased estimator):

$$\boxed{\hat{\Phi}_{xy}(\tau) = \frac{1}{N}\sum_{i=\tau}^{N} x(i-\tau)y(i)}$$

- Use large N to minimize bias! I.e. $\frac{N}{N-\tau} \rightarrow 1$
- Similar estimators can be derived for the covariance and correlation coefficient functions

Additional demo's

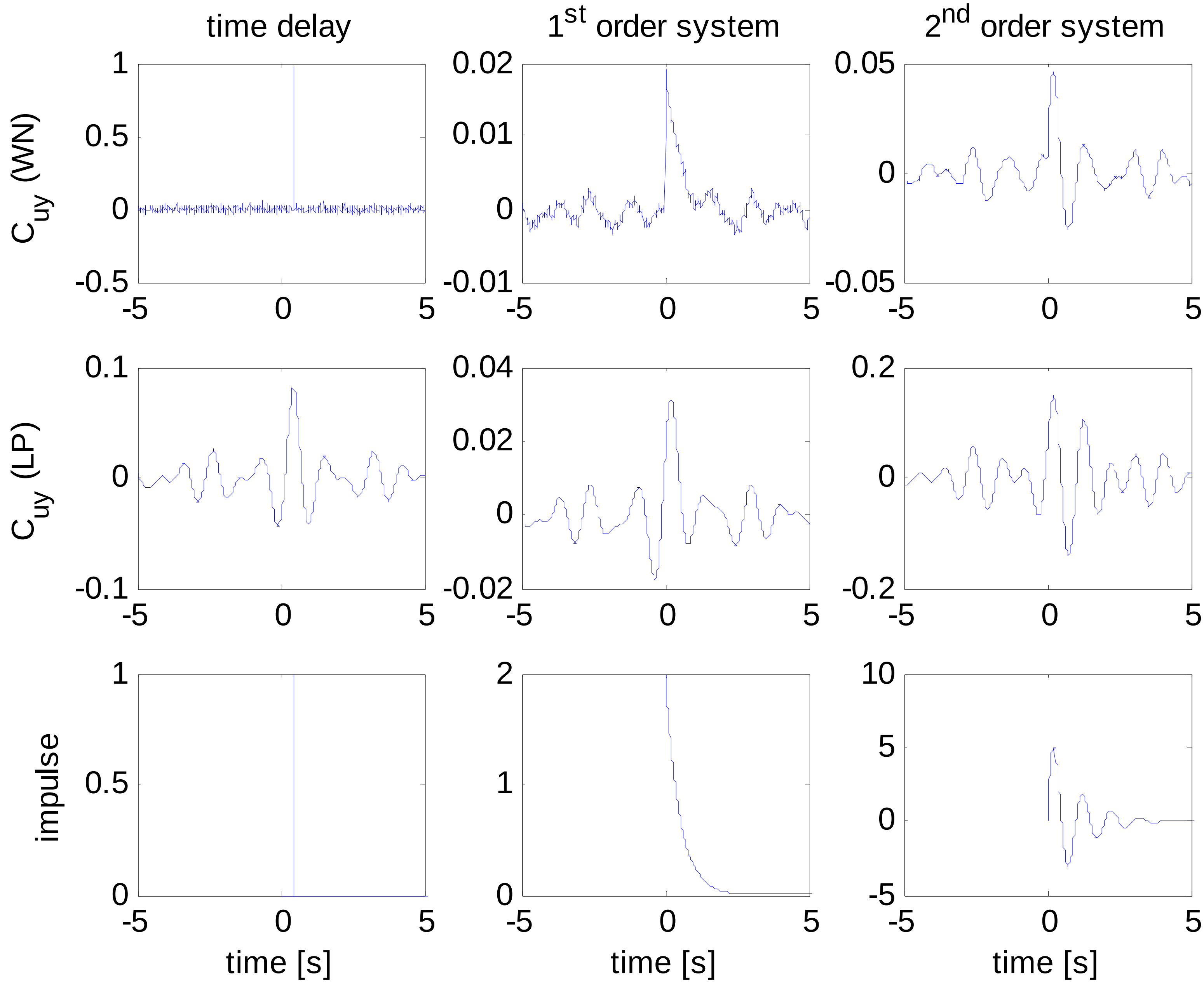
- Calculation of cross-covariance: Lec2_calculation_Cuy.m

$$\hat{\Phi}_{xy}(\tau) = \frac{1}{N}\sum_{i=\tau}^{N} x(i-\tau)y(i)$$

Cross-covariance of some basic systems

- Example systems (Matlab Demo: Lec2_example_systems.m):
  - Time delay
  - 1st order
  - 2nd order

- Input signals:
  - White noise (WN)
  - Low pass filtered noise (LP: 1/(0.5s+1))


time delay
1st order system
2nd order system
C_uy (WN)
C_uy (LP)
impulse
time [s]
time [s]
time [s]

Intermezzo: Fourier Transform

- Fourier Transform:

$$X(f) = \Im\left(x(t)\right) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

- Mapping between time-domain and frequency-domain
  - One-to-one mapping
  - Unique: inverse Fourier Transform exists
  - Linear technique

- Inverse Fourier Transform:

$$x(t) = \Im^{-1}\left(X(f)\right) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi tf}\, df$$

Fourier transformation

- y(t) is an arbitrary signal

$$Y(f) = \int y(t) e^{-j2\pi ft} dt$$

- Euler formula: $e^{jz} = \cos(z) + j\sin(z)$
  - symmetric part: cos(z)
  - anti-symmetric part: sin(z)

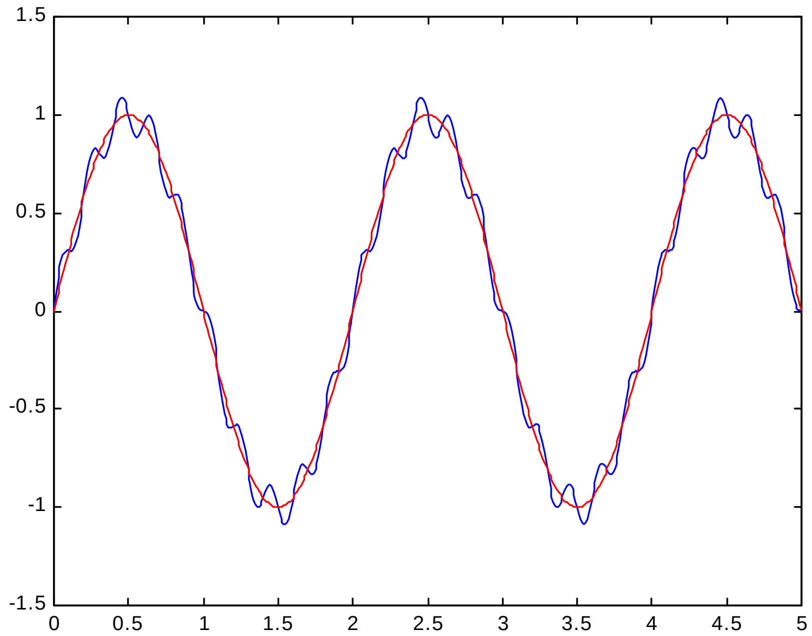
$$e^{-j2\pi ft} = re(e^{-j2\pi ft}) + im(e^{-j2\pi ft}) = \cos(2\pi ft) - j * \sin(2\pi ft)$$

Example Fourier Transformation

$$Y(f) = \int y(t) e^{-j2\pi ft} dt$$

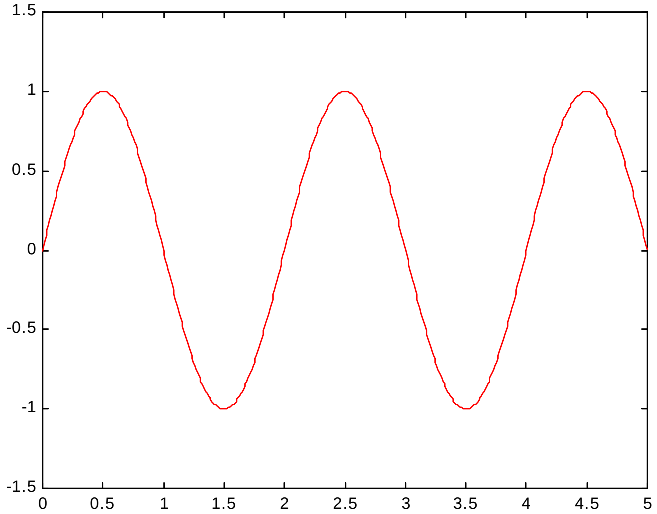
$$y(t) = \sin(0.5Hz) + \sin(5Hz)$$

Y(0.5Hz): 5 Hz signal will be averaged out

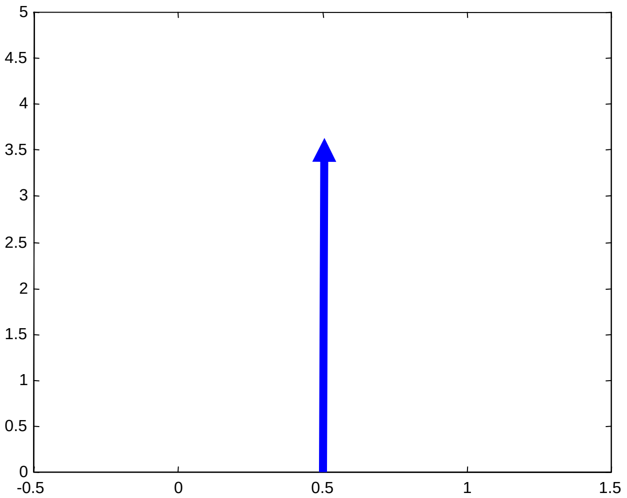


Example Fourier Transformation

$y(t) = \sin(0.5t)$



$Y(\omega) = \delta(\omega-0.5)$



Frequency Domain Expressions

- Discrete Fourier Transform:

$$U(f) = \Im\left(u(t)\right) = \sum_{t=1}^{N} u(t) e^{-j2\pi\frac{ft}{N}}$$

- where *f* takes values *0, 1, …, N-1* multiples of $\Delta f = \frac{1}{N\Delta t}$

- Inverse Fourier Transform:

$$u(t) = \Im^{-1}\left(U(f)\right) = \frac{1}{N}\sum_{f=1}^{N} U(f) e^{j2\pi\frac{ft}{N}}$$

Fourier transform of signals

- Discrete Fourier Transform (DFT)

$$u(k); \qquad k \in [0,1,\ldots,N-1]$$

$$U(r) = DFT\{u(k)\} = \sum_{k=0}^{N-1} u(k) e^{-j2\pi rk/N}$$

$$U(r); \qquad r \in [0,1,\ldots,N-1]$$

- DFT maps N real values in time domain to N complex values in frequency domain
- Double information?
  - DFT is symmetric U(-r)=U(r)*
  - Information for N/2 complex values

$$r \in \left[0,1,\ldots,\frac{N}{2}\right]$$

Power spectrum or auto-spectral density

- auto-spectral density S_{xx} is Fourier Transform of autocorrelation

$$S_{xx}(\omega) = \Im\left(\Phi_{xx}(\tau)\right) = \int_{-\infty}^{\infty} \Phi_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{xx}(f) = \Im\left(\Phi_{xx}(\tau)\right) = \int_{-\infty}^{\infty} \Phi_{xx}(\tau) e^{-2\pi jf\tau} d\tau$$

- properties:
  - Real values (no imaginary part)
  - Symmetry: $S_{xx}(f)=S_{xx}(-f)$ => only positive frequencies are analyzed
  - $C_{xx}(0) = \int S_{xx}(f)\, df = \sigma_x^2$
    Area under $S_{xx}(f)$ is equal to variance of signal (Parseval's theorem)

Cross-spectrum or Cross-spectral density

- cross-spectral density S_{xy}:

$$S_{xy}(f) = \Im\left(\Phi_{xy}(\tau)\right) = \int_{-\infty}^{\infty} \Phi_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

- properties:
  - Complex values
  - $S_{xy}^*(f)=S_{xy}(-f)$ => only positive frequencies are analyzed
  - $S_{xy}(f)$ describes the interdependency of signals x(t) and y(t) in frequency domain (gain and phase)
  - if $\Theta_{xy}(\tau) = 0$, then $S_{xy}(f) = 0$ for all frequencies

Estimators for the power spectrum

- Fourier transform of the autocorrelation function, the power spectrum:

$$\hat{S}_{uu}(f) = \sum_{\tau=0}^{N-1} \hat{\Phi}_{uu}(\tau) e^{-j2\pi \frac{f\tau}{N}}$$

- using the time average estimate: $\hat{\Phi}_{uu}(\tau) = \frac{1}{N}\sum_{i=\tau}^{N} u(i-\tau)u(i)$

and multiplication with $e^{-j2\pi \frac{(i-i)f}{N}} = e^0 = 1$

- gives: $\hat{S}_{uu}(f) = \frac{1}{N}\sum_{\tau=0}^{N-1} u(i-\tau) e^{j2\pi \frac{(i-\tau)f}{N}} \sum_{i=\tau}^{N} u(i) e^{-j2\pi \frac{if}{N}}$

$$= \frac{1}{N} U^*(n\Delta f) U(n\Delta f)$$

Indirect approach and direct approach give equal result!

* : complex conjugate:

$U^*(n\Delta f) = U(-n\Delta f)$

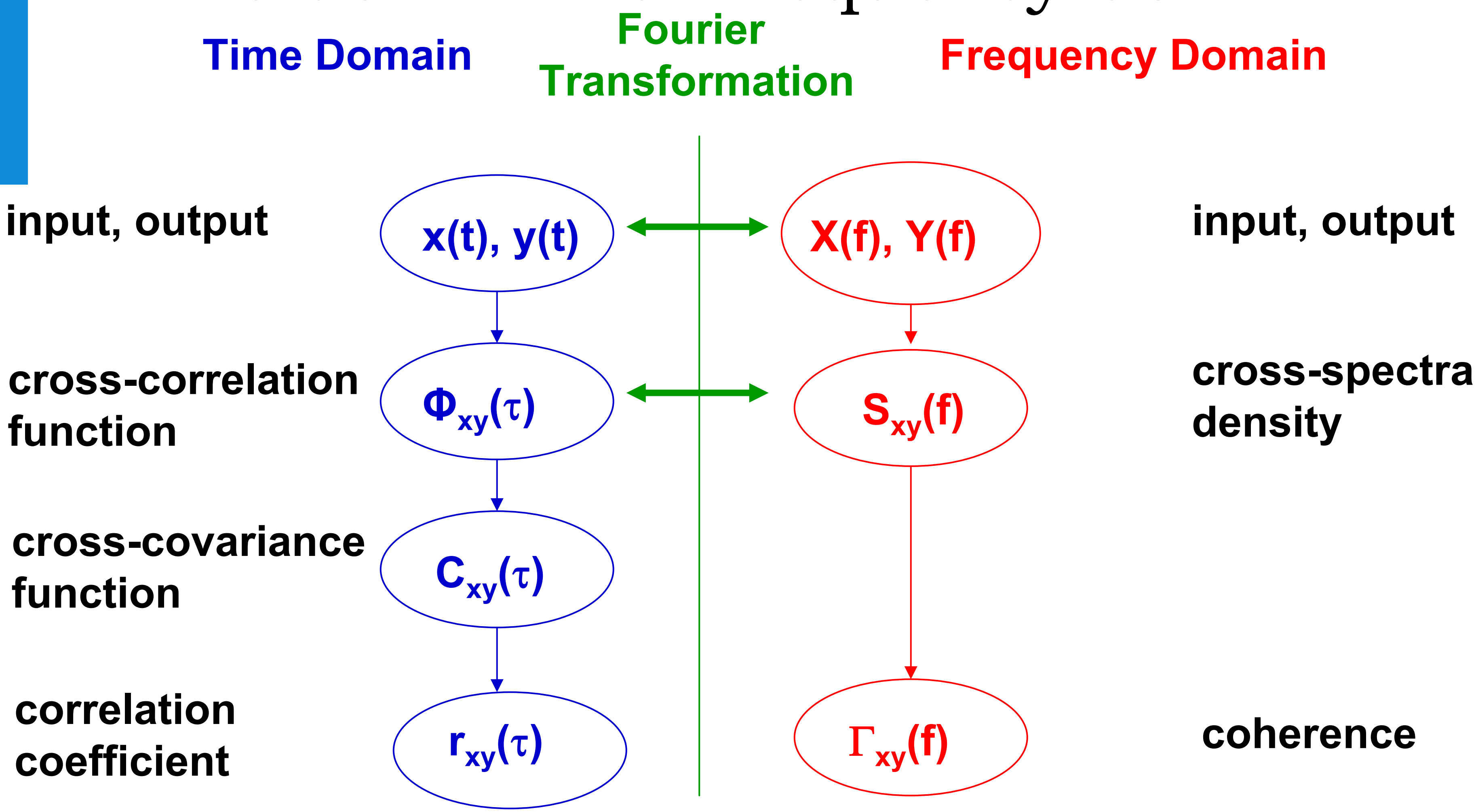
Estimator for the cross-spectrum

- Fourier transform of the cross-correlation function is called the cross-spectrum:

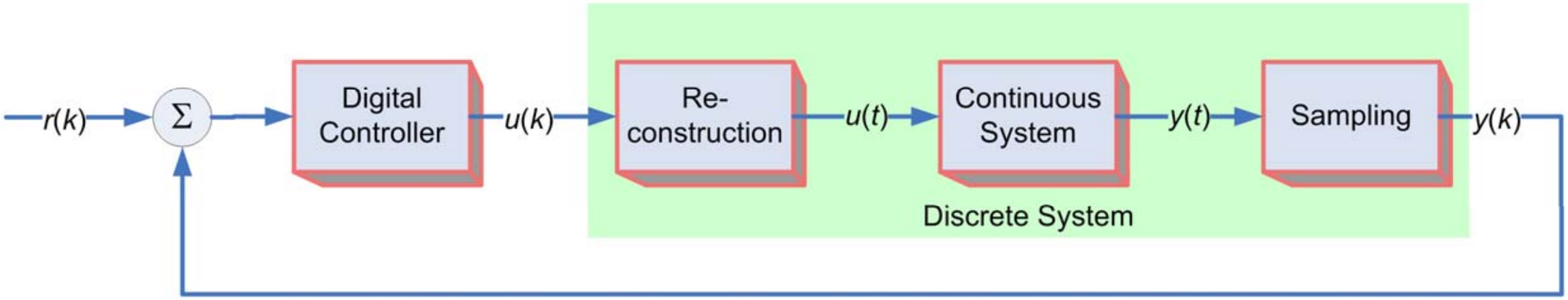
$$\hat{S}_{uy}(f) = \sum_{\tau=0}^{N-1} \hat{\Phi}_{uy}(\tau) e^{-j2\pi \frac{f\tau}{N}}$$

$$\boxed{\hat{S}_{uy}(f) = \frac{1}{N} U^*(n\Delta f) Y(n\Delta f)}$$

Time-domain vs. Frequency-domain



Discrete and Continuous Signals

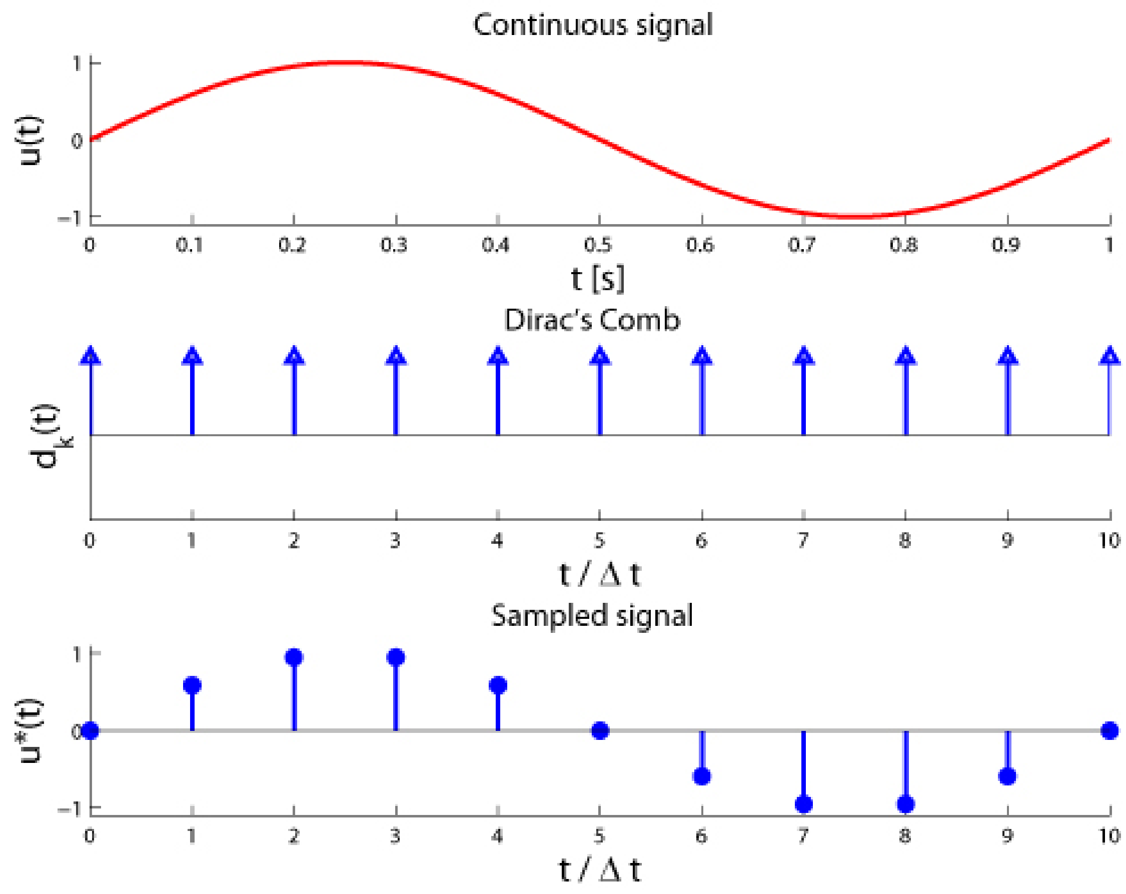


- Reconstruction
  - DA conversion
  - Distortion: Hold Circuits
- Sampling
  - AD conversion
  - Distortion: "Dirac Comb"

- Continuous system: Plant

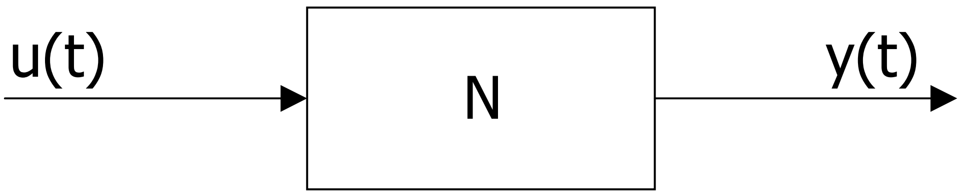
- Digital Controller: Plant performance enhancement

Sampling, Dirac's Comb



Models of Linear Systems

- System: $y(t) = N(u(t))$



- Linear systems obey both scaling and superposition property!

$$k_1 y_1(t) = N(k_1 u_1(t))$$

$$k_2 y_2(t) = N(k_2 u_2(t))$$

$$k_1 y_1(t) + k_2 y_2(t) = N(k_1 u_1(t) + k_2 u_2(t))$$

Time domain models

- Assume input is a pulse with:

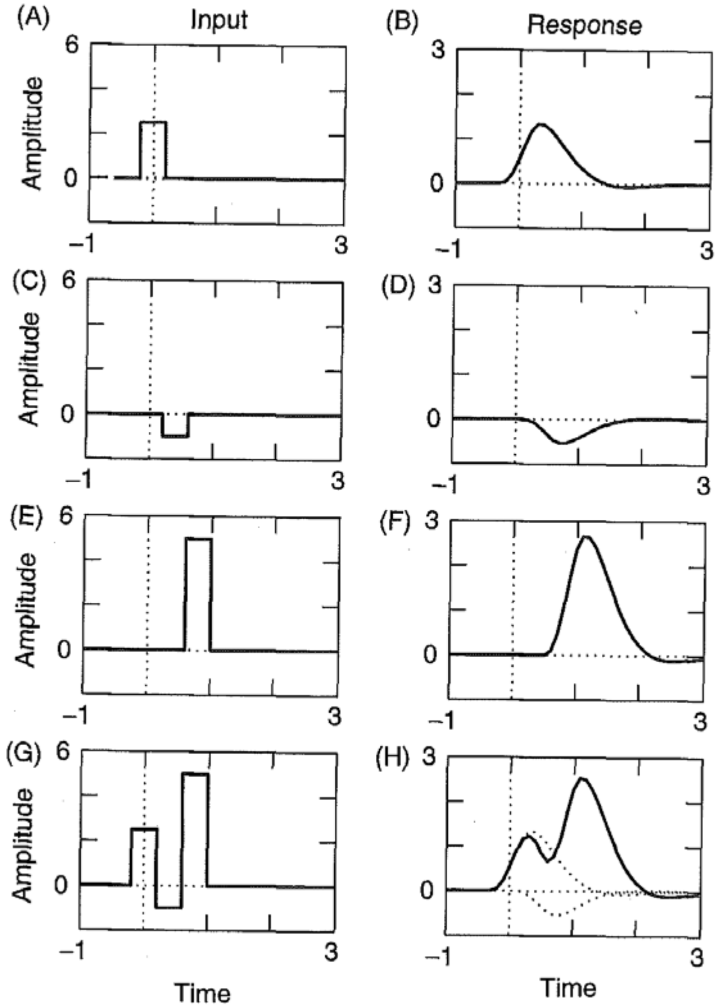
$$d\left(t,\Delta t\right)=\begin{cases}\left(\dfrac{1}{\Delta t}\right) & \text{for } |t|<\dfrac{\Delta t}{2}0 \\ 0 & \text{otherwise}\end{cases}$$

- Response of linear system N to pulse:

$$N\left(d\left(t,\Delta t\right)\right)=h\left(t,\Delta t\right)$$

- Assume u(t) is weighted sum of pulses:

$$u\left(t\right)=\sum_{k=-\infty}^{\infty}u_k d\left(t-k\Delta t,\Delta t\right)$$



Time domain models

- The output is: $y(t) = N(u(t))$

$$= \sum_{k=-\infty}^{\infty} u_k N\left(d\left(t - k\Delta t, \Delta t\right)\right)$$

$$= \sum_{k=-\infty}^{\infty} u_k h\left(t - k\Delta t, \Delta t\right)$$

- Limit case: unit-area pulse d(t,Δt) is an impulse: $\lim_{\Delta t \to 0} d\left(t, \Delta t\right) = \delta\left(t\right)$

- Then h(t) is the impulse response function (IRF) and the output the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h\left(\tau\right) u\left(t - \tau\right) d\tau$$

Time domain models

- Causal system: h(t)=0 for t<0
- Finite memory: h(t)=0 for t>T

$$y(t) = \int_{o}^{T} h(\tau)u(t-\tau)d\tau$$

- Discrete

$$y(t) = \sum_{\tau=0}^{T-1} h(\tau)u(t-\tau)\Delta\tau$$

Frequency domain models

- Time-domain: convolution integral $y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$
- Fourier transform: $Y(f) = \Im(y(t)) = \Im\left(\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau\right)$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau\right)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} h(\tau)\int_{-\infty}^{\infty} u(t-\tau)e^{-2\pi ft}dtd\tau$$

'Convolution in time domain is multiplication in frequency domain' (and vice-versa)

$$Y(f) = H(f)U(f)$$

Estimators in time domain

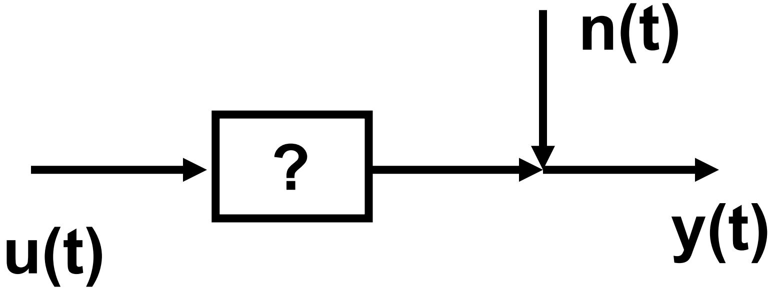
- Cross-covariance
  - When analyzing the dynamics of a dynamical system interested in variations of the signals around it mean (and not average value).

- Assuming signals with zero-mean

$$y(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(t')u(t-t')dt'$$

$$C_{uy}(\tau) = E\left\{u\left(t-\tau\right)y\left(t\right)\right\} = E\left\{\int_{-\infty}^{\infty} h(t')u(t-\tau)u(t-t')dt'\right\}$$

Basic identification with cross-covariance



$$y(t) = n(t) + \int h(t')u(t-t')\,dt'$$

multiply with $u(t-\tau)$: $u(t-\tau)\,y(t) = u(t-\tau)\,n(t) + \int h(t')u(t-\tau)u(t-t')\,dt'$

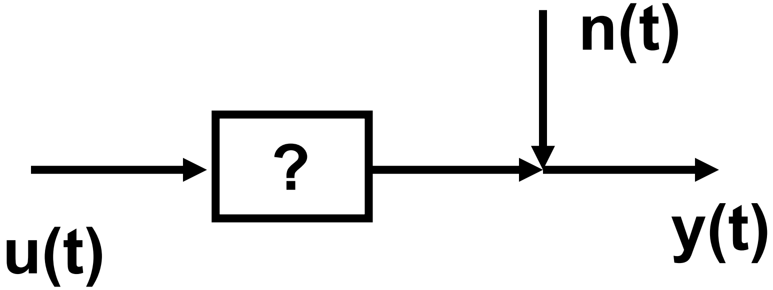
$$C_{uy}(\tau) = C_{un}(\tau) + \int h(t')C_{uu}(\tau - t')\,dt'$$

white noise: $C_{uu}(\tau) = 0 \text{ for } \tau \neq 0; \quad C_{uu}(0) = 1$

$$C_{uy}(\tau) = C_{un}(\tau) + h(\tau)$$

Other 'tricks' needed when u(t) is not white

Basic identification with spectral densities



$$y(t) = n(t) + \int h(t')u(t-t')\,dt'$$

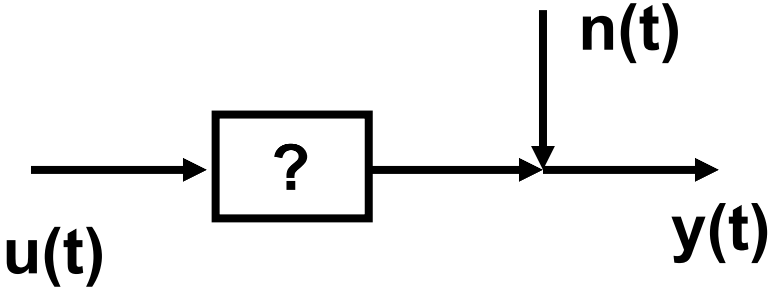
multiply with $u(t-\tau)$: $u(t-\tau)\,y(t) = u(t-\tau)\,n(t) + \int h(t')u(t-\tau)u(t-t')\,dt'$

$$C_{uy}(\tau) = C_{un}(\tau) + \int h(t')C_{uu}(\tau - t')\,dt'$$

Fourier transform: $S_{uy}(f) = S_{un}(f) + H(f)\,S_{uu}(f)$

if $S_{un}(f) = 0$: $H(f) = \dfrac{S_{uy}(f)}{S_{uu}(f)}$

Identification:
time-domain vs. frequency-domain



- Time domain
  - Unknown system: impulse response function (IRF) of h(t′) over limited time
  - Mostly: direct model parameterization (fit of physics model)

- Frequency domain
  - Unknown system: frequency response function (FRF) H(f) for number of frequencies

Example IRF & FRF of typical systems

- Time delay
- First order system
- Second order system
- Unstable system

- (try Matlab!)

Book: Westwick & Kearney

- Lecture 1: signals
  - Chapter 1, all
  - Chapter 2, sec. 2.1 – 2.3.1
- Lecture 2: Correlation functions in time & frequency domain
  - Chapter 2, sec. 2.3.2 – 2.3.4
  - Chapter 3, sec. 3.1 – 3.2
- Lecture 3: Estimators for impulse & frequency response functions
  - Chapter 5, sec. 5.1 – 5.3

Next week: lecture 3

- Lecture 3:
  - Estimation of IRF and FRF
  - Estimator for coherence
  - Open loop and closed loop system identification